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P

TEXT BOOK

OF

MECHANICAL PHILOSOPHY.

FOR THE USE OF STUDENTS.

BY THE

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PART I.

MECHANICS.

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PREFACE.

THE recent Statute concerning the Examinations for the Degree of B.A. has established a School of Natural Science, and has enacted, that no Student should receive a "Testamur" in this School, unless he exhibits an acquaintance with the elements of at least two out of the three great divisions of Natural Science, viz. Mechanical Philosophy, Chemistry, and Physiology; and *also* with some Physical Science dependent on Mechanical Philosophy. The words of the Statute are,

"In hac Schola nullus Candidatus Testimonio dignus habeatur, nisi Principiis Philosophiæ Mechanicæ, Chemiæ et Physiologiæ vel trium harum Scientiæ Naturalis partium duabus saltem

sit imbutus et insuper in Disciplina quadam Physica quæ ex Philosophia Mechanica pendet, ita profecerit, ut Examinatoribus satisfaciat."

The Statute further enacts, that for Honorary Distinction the Candidate must have a more extended knowledge of all the three divisions above mentioned, and an exact acquaintance with some one or more of the subordinate Sciences.

The accompanying volume has been commenced in order to supply the Student with a Text Book for the elements of one of the three great divisions of Science mentioned in the Statute. It is shewn in the Introduction in what manner all the Natural Sciences may be brought under those three heads: although at the same time it is not supposed that this or any other arrangement is complete or in all respects satisfactory. The remarks which are there added will point out the impossibility of making any such classification exact. In fact, the three Sciences which are called Mechanical Philosophy, Chemical Philosophy, and Physiology, ought to be regarded rather as lying at the root of the Physical Sciences than as containing them

all. For this reason they are called by Dr. Daubeny, (in the Pamphlet referred to,) the *Primary* Sciences, the other Sciences being *special* or *subordinate*. The elements of these three Primary Sciences ought to form part of every complete system of education. The classification is given, that the Student may observe the relation of any special Science which he may study to the other departments of knowledge.

The Part now published contains only so much of the elements of Mechanical Philosophy as is usually expressed under the title of *Mechanics*. It pretends to no more than that which the title assumes, viz. to be a Text Book, which the Student must fill up by reading or attendance at Lectures. It may be found perhaps that the subject has been carried farther than can be required of Candidates for an ordinary Degree, but the Book is designed for Candidates for Honours, as well as for those who desire only to obtain a "*Testamur*." In order to mark distinctly the important conclusions of the Science, the form of enunciation by Propositions printed in a different

type, has been for the most part observed. Mathematical reasoning and Mathematical formulæ have been avoided, except in cases where it would not have been possible to have exhibited conclusions without much circumlocution, and consequently with a loss of clearness. No more Mathematics, however, are required for the whole volume than a knowledge of the early part of Euclid, the common symbols of Algebra, the definitions of Trigonometry, and the chief property of the Parabola.

Part II. containing the elements of Hydrostatics and Pneumatics, is in preparation, and will shortly be published: and it will be followed by other Parts, containing the elements of the other Sciences, which are usually classed under the head of Mechanical Philosophy.

October 18, 1851.

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INTRODUCTION.

DIVISION OF THE PHYSICAL SCIENCES.

1. PHYSICAL Science, in its widest acceptation, includes every investigation into the nature, properties, and changes of the various bodies of the material universe: it examines every form of matter which exists in the creation of God, every thing which is the subject of change: the changes themselves, the causes of those changes, and the laws by which those changes are regulated.

2. This general definition leads at once to two great divisions. The investigation may be with a view to classification and arrangement; it may be merely a science of observation, in which case it is properly called *Natural History:* or it may enquire into the causes and the laws of

the changes which the forms of matter undergo, and thus may be that which is more properly called *Physical Science.*

3. But material bodies may be distributed into two classes; viz. *organized* (or animate) bodies, and *unorganized* (or inanimate) bodies: and thus we divide Physical Science into the two branches of *Physiology* and *Physics*, in the restricted sense in which the term is frequently employed.

4. Physiology, therefore, will comprehend every enquiry into the condition, properties, and changes of the bodies of the *animal* and *vegetable* kingdoms. The substances which belong to those kingdoms have all peculiar organisms adapted for appropriate functions; and they have a principle of vitality by which they are clearly distinguished from the other substances in nature; this principle of life is retained only for a time, when they sink into the class of inanimate bodies, but their organizations are still the subject of Physiological research.

5. Physics, in the more limited sense, investi-

gates the properties of *unorganized* bodies, the forces which act upon them, and the changes they undergo: but these properties and forces and changes may be either *internal* or *external;* and if we are engaged on the *internal* properties and changes, we are within the province of *Chemical Philosophy;* but if on the *external*, we are in the province of *Mechanical Philosophy*.

6. *Chemical Philosophy* may then be said to treat of the *individual* properties of unorganized bodies, by which, as regards their material constitution, they may be distinguished from one another, to investigate the transformations which take place within the body, and by which the very substance of the body may seem altered; and also to enquire into the laws by which those transformations are regulated.

7. *Mechanical Philosophy*, on the other hand, teaches the *general* properties of unorganized bodies, the forces which act upon them, the laws which they obey, the results thus produced, and all those external changes which leave the substance of the body unaffected.

8. All the Physical Sciences may be brought under one or the other of the three great divisions[a], *Physiology*, *Chemical Philosophy*, and *Mechanical Philosophy:* but it may be observed, that a science bearing one and the same name, may come under a different head, according to the principle and method of investigation. For instance, Chemistry is naturally placed under the department of Chemical Philosophy, but when applied to animal and vegetable phænomena, it is strictly Physiological. Thus also Geology may be regarded in one sense as Physiological; it is Mechanical when it assumes the name of Dynamic Geology, and it is a branch of Natural History when it merely tabulates, records, or arranges its observations.

9. Mechanical Philosophy is indeed essentially mathematical, and for its accurate study it requires the aid of mathematical symbols and

[a] This division accords with that given in a pamphlet, entitled, "Brief Remarks on the Correlation of the Natural Sciences, by Charles Daubeny, M.D. Professor of Chemistry and Botany. Oxford, 1848."

mathematical reasonings; but for the purpose of elementary instruction, we may deduce its principles from experience and observation: we may present the conclusions which are deduced by mathematical investigations, and illustrate them by experiments and facts. And further than this, by proper arrangement of experiments we may make advances in the science, and so point out the course which should be taken by the stricter scrutiny of mathematical enquiry. It is this which is the office of Experimental Philosophy.

10. Material substances are commonly arranged in two classes. Those of the first class are called *ponderable* bodies, or sometimes *coercible*, or sometimes *tangible*: those of the other class are *imponderable*, or *incoercible*, or *intangible*. Ponderable bodies again are subdivided, as *solid*, *liquid*, or *aeriform*[b]; and, according to these distinctions, Mechanical Philosophy comprehends

[b] The subdivision anticipates the remarks made in the following chapter, but it has been thought best to introduce it at once, in order to complete the classification.

several sciences, as may be seen in the following table.

Mechanical Philosophy, as treating of Ponderable Bodies: solid = Mechanics properly so called[c]; liquid = Hydrostatics; aeriform = Pneumatics.

Mechanical Philosophy, as treating of Imponderables: Light; Heat; Electricity.

11. According to this division, the great Science of Astronomy is a branch of Mechanics, as the term *Celestial Mechanics* clearly indicates; yet again, when it is a mere science of observation, its place is rather under Natural History. Astronomy is, however, of such importance and interest, that it requires a distinct discussion; but it will be necessary for the student of Mechanical Philosophy to apply the principles of Mechanics to the heavenly bodies, and to obtain illustrations and instances from their motions.

[c] Sir J. Herschel, in his elegant Discourse on the Study of Natural Philosophy, (Lardner's Cabinet Cyclopædia,) suggests, that what is commonly called Mechanics, might without impropriety be called *Stereostatics*.

12. The Science of Acoustics again may be regarded as a branch of Pneumatics, inasmuch as sound is ordinarily communicated by the motion of air. Solid bodies convey sound as rapidly as air, but it will be convenient nevertheless to investigate the phænomena of sound after we have discussed the Science of Pneumatics.

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BOOK I.

CHAP. I.

EMENTS OF THE PHYSICS OF PONDERABLE OR COERCIBLE BODIES.

GENERAL PROPERTIES.

13. THE properties of bodies are either *essential* and general, or non-*essential* and peculiar. The former are common to all bodies, the latter give us distinctions between bodies, and are of great value on account of their applications.

14. The essential properties of bodies are *Extension*, *Impenetrability*, and *Divisibility*; to which may be added *Inertia* and *Gravitation*.

15. *Extension* is that property by which a body has a limited form: by it the volume or size and the figure of the body is determined.

The extension of a body is expressed by the three dimensions of length, breadth, and thickness. We speak also of *linear* magnitude, *superficial* magnitude, *solid* magnitude.

16. *Impenetrability* is the property by which two bodies cannot occupy the same position in space at the same time; a second body must thrust out the first before it can take its place.

It might be imagined that this property belonged only to solid bodies, but closer observation shews that is necessary and common to all, whether solid, liquid, or aeriform. The apparent examples of *penetration* are only examples of displacement.

The cases of mixture, in which two different liquids when combined do not produce a volume equal to the sum of the two, are *not* exceptions to this property.

17. *Divisibility*, or the capability of being subdivided, is or is not an essential property, according as there is a limit or no to the subdivision. As far as our experiments go, there is no practical limit to the divisibility of matter. Some bodies are more easily divided than others, but in every form, matter may be reduced to particles of the most unlimited minuteness, and each of

these minute particles has properties and qualities similar to the larger mass from which it has been obtained.

18. There is however a limit to this divisibility on the most received theories with regard to the constitution of matter. Various hypotheses have been brought forward at different times to explain the formation of bodies and the different states of matter; but it seems now most probable, that all bodies are made up of certain minute particles or atoms of determinate magnitude and figure. These atoms are of different forms in different substances, and are endued with different forces, and hence give rise to different properties, as well as to the different forms of bodies. We are not indeed able to arrive at these atoms by subdivision, nor can we exhibit them to the eye; yet there are many phænomena which make their existence probable, and which are easily explained by the hypothesis.

19. According to this assumption, the forces with which these atoms or molecules are endued, bring the atoms into a certain fixed position with regard to each other, and there hold them together into one body. This power or force with which the particles are held in this stable condition is

called the *force of cohesion.* It is measured by the resistance it offers to the mechanical subdivision of the body.

20. The atoms are endued also with another force which holds this attractive force, or force of cohesion, in check, and prevents actual contact between the particles, or at any rate in all positions alike. This force is a *force of repulsion,* and by the relations which exist between this force and the force of cohesion or attraction, the different aggregate forms of bodies are presented. If the attraction predominates, we have a solid state; if the repulsion prevails, the gaseous forms result; whereas when neither force is in excess, the body is liquid.

These conditions may be exhibited in the following formulæ:

$$A > R, \text{ solid},$$
$$A = R, \text{ liquid},$$
$$A < R, \text{ gaseous},$$

where A represents the attractive force, and R the repulsive.

21. The force of cohesion in the atoms is not the same in all directions, in all cases. This is shewn in the beautiful phenomena of Crystallization.

22. *Cohesion* must be distinguished from

Adhesion. This latter is the same as surface attraction, and is exerted when two smooth surfaces of the same or different substances adhere, or when a liquid wets a solid, &c.

23. Cohesion is overcome in most cases by heat, which seems to increase or produce the force of repulsion. This is seen in the melting of a solid, which first becomes a liquid, and then with greater heat may assume a gaseous form, as in the case of ice, first becoming water, and then steam.

24. The ultimate atoms are however indestructible. Although by heat the body may be subdivided and disappears to our sight, yet the atoms may be brought together again, and are none of them lost. Nothing short of the power of the Creator can destroy them.

25. It must be observed, however, that whether this supposition about the constitution of matter be true or no, the principles of Mechanical Science are independent of all hypothesis, and rest upon *observed facts*, and on strict reasoning from them.

26. *Inertia*, or the inactivity of matter, is a quality of matter which is of the utmost importance in all mechanical enquiries. It is one, however,

which has given rise to many erroneous conceptions. Inertia is a merely negative term, implying that matter is unable to put itself into motion, or spontaneously to change its motion. Whatever be its state of rest or motion, in that state the matter must continue, unless some external agency is brought to bear upon it.

In saying then that matter is *inert*, we merely state that it has no vitality or spontaneity.

The consequences of this important quality will be traced in Book III.

27. One other property may be regarded as common to all bodies with which we are acquainted in this our system, and that is the great property of *Gravitation.* By virtue of this property or power, impressed by the will of the Creator, every substance has a tendency to draw towards itself every other substance, which in its turn is attracted by the other. This power pervades the universe as far as our observations extend. In consequence of this, all bodies near or upon the surface of this our planet exhibit a tendency to approach it, and to move towards its centre. Hence all bodies if unsupported fall towards the earth, and they exert a pressure on any thing which prevents their thus falling. From

this, all bodies possess that property which we call *weight*. This is a property of terrestrial bodies, of which we shall trace the consequences hereafter.

CHAP. II.

NON-ESSENTIAL PROPERTIES OF MATTER, OR SPECIFIC PROPERTIES.

28. THE component particles of bodies are not, as we have observed, (art. 20.) in actual contact. Numerous experiments establish this fact, especially those which shew that every body can be compressed into smaller dimensions. The repulsive force which exists in the atoms is the cause of certain intervening spaces between them, and when this force is exerted more intensely in one direction than in another, these intervening spaces are disposed with a certain regularity, and produce what are called *pores*.

29. *Porosity* is almost a general property of all matter, but the term is strictly employed to denote the existence of regular openings or intervening spaces, such as are seen in wood, cane, and bone, &c. The pores when extending in one direction, as in the instances named, are properly called *longitudinal pores*.

The larger openings or cavities, such as are seen in sponge, are *not* pores; they are like the cells of a honeycomb.

The densest substances, such as gold, minerals, &c. have pores, and these are commonly filled with air.

The process of filtration in the arts depends on the existence of these minute interstices called *pores*.

30. *Compressibility*, like porosity, is perhaps an universal property of matter. It is a proof (art. 28.) of the existence of pores, and conversely if there are pores, the body may be reduced in dimensions by any force which can overcome the repulsion which exists between the atoms, and hence gives rise to pores.

Those bodies in which there are longitudinal pores, are more easily compressed laterally.

All forms of matter are compressible to some extent. The most *solid* metals may be reduced in bulk by hammering.

The experiments of Canton, OErsted, and Perkins, have shewn, that *liquids* may be compressed by the application of a sufficient force, while a very small one is sufficient to reduce the dimensions of *aeriform* bodies.

Dilatability is a property the opposite to compressibility. It may be exhibited in many instances by the application of mechanical force, but the great agent for exhibiting this property is *heat*. By heat the dimensions of almost every substance is altered; probably every substance is thus affected by heat.

Metals expand by heat, though not all equally: the applications of the expansions of metals by heat and their contraction by cooling are very numerous, and the force both of expansion and contraction is very great.

Liquids alter their dimensions by heat, and most liquids expand in proportion to additions of temperature. Water is however an exception. It contracts as the temperature is lowered up to a certain point, but before it reaches the freezing temperature it expands. There is also another break in the continuity of the law of expansion just as it comes to the boiling point[a].

Aeriform bodies expand rapidly on a very slight elevation of temperature.

33. *Elasticity* is a property possessed in very different degrees by different bodies, but it is

[a] See on this subject Whewell's Bridgewater Treatise, book i. chap. 9.

important to distinguish between this and the last property. Elasticity is the property by virtue of which a body having been compressed recovers its former dimensions when the compressing force is removed.

Inelastic bodies are those which do not recover their shape after having been compressed, such as wet clay.

The most elastic of solid bodies are ivory, glass, steel.

Liquids are also elastic, but they admit of compression only by powerful mechanical agents, and the property of elasticity is seldom called into play.

Gases and airs are *extensively* elastic.

It will appear less singular, as remarked by Mosely[b], that aeriform bodies, liquids, and solids, should possess, in common, the property of elasticity, if we consider that these are but different forms under which the same body may exist, subject to different conditions of heat. Steam, for instance, condenses into *liquid* water, a certain abstraction being made of its heat: and this water, by another reduction of its temperature, becomes *solid* ice. And so, in the

[b] Illustrations of Mechanics, page 35.

opposite direction, a *solid* metal becomes a *liquid* by the application of heat, and a yet greater intensity of heat *volatilises* it.

34. The property of elasticity is presented in a somewhat different view, when a wire of any substance is twisted, and its tendency to recover its natural state is observed. Elasticity thus observed is called *elasticity of Torsion.* The law of this force has been carefully estimated, and it is found to be always proportional to the angle through which the body has been twisted. This force exists not only in wires of metal, but more or less in all substances.

Coulomb has applied this elasticity to the measuring of forces too minute to be estimated by the ordinary methods. His instrument is called the Torsion Balance. It was by a balance of this kind that Cavendish detected and measured the force of gravitation.

35. *Hardness* is a property of bodies closely connected with elasticity. It is estimated by the force by which the component atoms or molecules resist separation or change of relative position.

Hardness is a modification or peculiar effect of the force of cohesion.

It is the property possessed by most stones,

several metals, and various kind of woods, and by which they resist a considerable external force either of pressure or impact.

The diamond is considered the hardest known substance: it will cut or scratch all other bodies.

Hardness has no connection with density, or the number of molecules in a given volume. Mercury, for example, is *dense*, but not *hard*. Glass is *harder* than gold or silver, but not so *dense*.

36. *Softness* is a term employed to express the opposite property to hardness. Those bodies are called soft which change their form by the application of any small force, no disruption of the molecules taking place, and the form impressed being retained after the cessation of the force.

Moist clay, tallow, butter, &c. are familiar examples of soft bodies.

The different states into which steel may be brought, afford beautiful illustrations of the properties of softness and hardness, and are of the greatest value in the arts.

37. Brittleness is a property possessed by some hard bodies. The molecules of *brittle* bodies

cohere strongly, but the range of the force of cohesion is so small, that if the molecules are separated in the slightest degree, a total disruption takes place. In some very brittle bodies the separation of one or more of the molecules causes a complete dissolution of the whole body.

Pottery presents familiar instances of hard and brittle bodies. Steel when highly tempered is also brittle.

Unannealed glass, (and especially a Bologna flask and Prince Rupert's drops,) are examples of very brittle substances.

The process of annealing, by allowing the substances to cool slowly, takes away the brittleness.

38. *Malleability* expresses the property by which a body may be extended in all directions by impact, the length and breadth may be increased or diminished, but the continuity is not broken. In malleable bodies, the molecules must cohere equally in all directions.

Several of the metals are very malleable, and are hence of great value in the arts. The most malleable of the metals are gold, silver, iron, and copper: but the malleability is much influenced

by temperature. The extreme malleability of gold is exemplified in *gold beating*.

39. *Ductility* is the property by virtue of which a substance may be drawn out into a wire. In ductile bodies the molecules would appear to adhere by poles, or in certain definite relative positions, but not in all positions alike.

It is not the same as malleability, for the most malleable metals are not the most ductile. Platinum, for instance, is the most ductile, then silver, iron, copper, and gold. Dr. Wollaston obtained platinum wire $\frac{1}{30,000}$th part of an inch in diameter.

Melted glass is very ductile, and the fibres have been applied for making ornamental damasks. More than 100 threads of the glass are frequently used to make one thread for the loom. It has been observed, that the minute fibres of glass are always of the same form as the bar of glass from which they are drawn.

40. *Pliancy* is the property by which a substance may be bent without being broken. It is generally applied to cases where the longitudinal dimensions far exceed the lateral, and results from another modification of the force of cohesion.

It is the property with which we are familiar

in animal and vegetable fibre, and also in thin wires of metals.

41. It may be here remarkable, that the various modifications of solidity, such as hardness, brittleness, &c. are by no means properties that invariably admit of strongly defined lines of demarcation, nor are they unsusceptible of variation in the same body. Heat exerts a great influence on all these properties.

42. *Tenacity* is a term usually employed to express the force by which a body resists separation by extension in the direction of its length. This force must not be confounded from that which gives the property of brittleness. Some bodies are very tenacious, which at the same time are very brittle. Glass, for example, is very brittle, but a rod of glass may be made, when suspended in a vertical position, to support a very great weight.

43. The whole question of the relative strength of materials, not only in the direction of their length but as offering resistance in every direction, is of the greatest importance. The following points are those which must be principally determined.

a. The *absolute strength* or resistance to tension,

i. e. the resistance which a rod, or wire, or cord, fastened at one end, offers to a force acting at the other end in the direction of the body's length, and *tending to tear it asunder*, i. e. *tenacity*, as above stated.

β. The *lateral strength*, or the force with which a beam or bar, fastened or supported at one or both ends, resists *rupture*, the stress being applied perpendicularly to the beam.

γ. The *resistance to compression*, or the force which a body opposes to a weight applied in the direction of its length, with a tendency either to *crush* the body, or by *deflection* to break it.

The weight required to effect the particular kind of rupture is in every case the measure of the strength.

Experiments have been conducted with great care to determine the strength of various substances in each of these particulars, and tables are given in works on the subject: such as,

Barlow's Treatise on the Strength of Timber, &c. (1837.)

Rennie, Philosophical Transactions, 1818.

Tredgold, ditto, 1824.

See also the results of the observations of Eaton Hodgkinson, Esq. and W. Fairbairn, Esq. as

published in the Manchester Memoirs, vols. iv. and v. and in the Reports of the British Association for 1837, and 1842.

Useful Tables are also given in Eytelwein's Handbuch der Statik, &c.

BOOK II.

STATICS.

CHAP. I.

THE COMPOSITION AND RESOLUTION OF FORCES APPLIED TO THE SAME POINT.

44. We now enter on that branch of Mechanical Philosophy which is usually called Mechanics, i. e. that Science which investigates the effects of forces acting upon portions of matter, and more particularly on matter in the form of a solid.

The properties of bodies, hitherto considered, are the consequences of the forces which one molecule exerts upon another. The effects now to be discussed depend on forces altogether *external* to the bodies acted upon.

45. The existence of a force is known by its effects, and in all cases those effects are the subject of investigation. The source and origin or nature of the forces may be various, but the effects alone are capable of being estimated.

46. Force is usually said to be that which produces or tends to produce motion or change of motion in any body: sometimes it is said to be that which produces pressure or motion. Either of these definitions of force, if they may be called definitions, come to the same point: and either lead to the ordinary division of the Science into the two branches of Statics and Dynamics, according as rest or motion is the result of the action of the force or forces.

47. In order to estimate the effects of a force, three points must be considered.

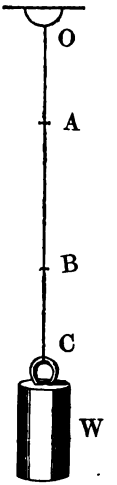
First, *the point of application*, or the point of the body or system of bodies to which the force is applied.

Secondly, its *direction*, or the line in which it tends to cause motion.

Thirdly, its *intensity* or *magnitude*.

48. With regard to the point of application of a force, it may at once be remarked, that it may be conceived to be shifted at pleasure along the line of direction. Experience shews that the effect of the force is the same at whatever point in the line of its direction it is applied, provided only that it can be made to act upon the body without introducing any contrivance to modify

its action. If, for example, a weight W (fig.) be suspended by a string from a point O, the strain upon the point is the same whether it be suspended from the point A, or B, or C. In this case the weight of the string is the same in all the cases; but if longer strings are introduced in order to shift the point of application along the line of direction, the weight of the string is a new force, and must be taken into account. Hence, in theory, we often introduce imaginary strings or rods without weights, &c. The weights of these can afterwards be estimated separately.



49. The *magnitude* of a force is very commonly expressed by weights, and weight is a force with which we are most familiar. This enables us to compare the relative magnitude or ratio of any number of forces with accuracy and simplicity. We have only to select some unit of weight as a standard of comparison, whether it be a grain, or a pound, or a ton, &c.

50. Forces may also be conveniently repre-

sented geometrically by straight lines, and this mode of representation exhibits very clearly the three points which should be observed. The beginning or origin of the line shews the point of application, the direction of the line shews the direction of the force, and the length gives the magnitude. It is only necessary to fix on an unit of length, and then measure off as many units of length as the given force would contain units of weight according to the last mode of estimation.

51. PROPOSITION. *Two or more forces acting together on the same point of a body in the same direction produce an effect equal to their sum, and the forces may be regarded as equivalent to a single force which is equal to them all together, acting in that direction.*

This is so self-evident, that it scarcely needs illustration or demonstration.

The single force which would thus produce the same effect as the several forces together is called the *Resultant* of these forces.

52. PROPOSITION. *Two forces acting together on the same point in exactly opposite directions produce an effect equal to their difference, and*

they may be regarded as equivalent to a single force equal to this difference, acting in the direction of the greater of the two.

This is also self-evident.

53. Hence it follows, that if the two forces are equal, the point will remain at rest; though it may be regarded almost as an axiom, that a point acted on only by two exactly equal forces in opposite directions cannot move in either direction or in any.

These two Propositions may be comprehended in one, which may be thus enumerated.

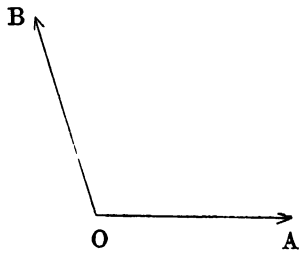
54. PROPOSITION. *If any number of forces act upon the same point, some in one direction, and the others in the direction exactly opposite, the* RESULTANT *of such a combination of forces is found by taking the difference between the sum of all the forces acting in one direction, and the sum of those acting in the other direction, and this resultant must be in the direction in which the sum of the forces is the greater.*

The several forces which thus combine to produce a resultant are called the *component* forces, and the terms resultant and component are obviously *correlative.* The resultant is mechanically equal to

the combination of its components, and the components are mechanically equal to the resultant: and in all mechanical investigations, the components may be substituted for the resultant, or the resultant for the components, without in any way changing the condition of the body on which the forces act.

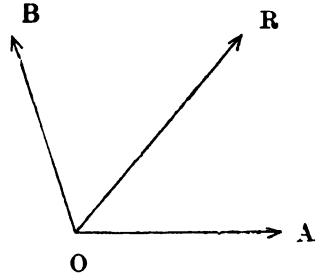
55. PROPOSITION. *If two forces act together on the same point, but are not in opposite directions, then whatever be the ratio of the forces, motion will ensue, or the point cannot remain at rest:* IN OTHER WORDS, *if two forces act on a point, and if their directions are divergent or inclined at an angle to one another, they cannot neutralize each other.*

Thus if OA, OB, (fig.) represent two forces acting on the point O, and making with one another any angle, as AOB, the point O will not remain at rest.



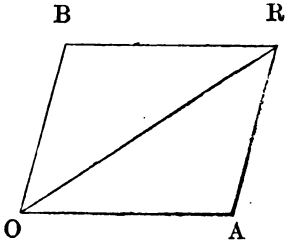
It is evident, that in this case the point O will have a motion in some direction within the

angle AOB, or the combined effect of the two will be in some line beginning at O, and within the angle, as OR, (fig.) the position of this line and its magnitude is however determined in the following important Proposition.



56. PROPOSITION. *If two forces acting on a point be represented in magnitude and direction by two lines drawn through that point, then the resultant of such forces will be represented in magnitude and direction by the diagonal of the parallelogram, of which the two lines are the adjacent sides.*

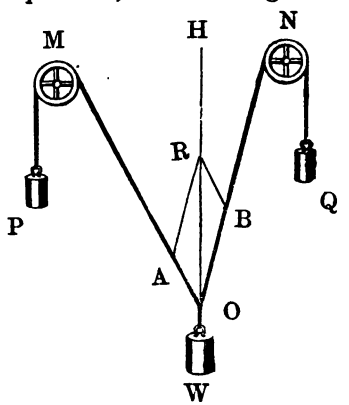
Thus if OA, OB, (fig.) represent the two forces, and the parallelogram OARB be completed, the diagonal OR will represent the resultant, and will singly produce an effect equal to the



combined effect of the two forces represented by OA, OB.

This very important proposition, which is known by the name of the Parallelogram of Forces, may be demonstrated by the help of mathematical reasoning and mathematical formulæ. The following experimental proof is more suited to the present work.

Let P and Q be two weights attached to the ends of a perfectly flexible cord, (the weight of which is so trifling that it may be neglected,) and let the string be passed over two pulleys M, N fixed in a vertical plane. Let another cord be fastened to this at a point O, a third weight W being suspended from it, (as in fig.) This weight W will draw the cord into some such position as in the figure, and after a few oscillations it will settle itself, and come to rest. Now it is



evident that the point O is solicited by three forces, first by the weight P, which in consequence of the pulley acts upon O in the direction OM: secondly, by the weight Q acting in the direction ON: and thirdly, by the weight W acting in the direction OW: and since the point O is at rest by the action of these three weights or forces, any two of them, as, for instance, those in OM, and ON, must together produce an effect which neutralizes or equilibrates the effect of the third in OW.

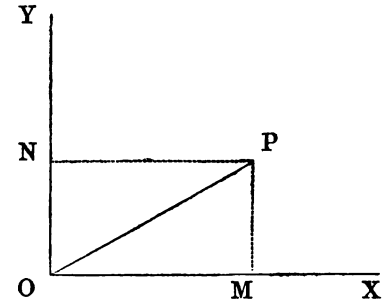
Again, it is evident that the weight W would be neutralized by an equal weight acting up at O in the direction OH opposite to the direction OW. Since then the force or weight in OW is really equilibrated by the two acting in OM and ON, these two must together produce an effect which acts on the point O in the direction OH. If now we were to measure on OH a length OR containing as many units of length as there are units of weight in W, and through the point R thus found draw OA, OB respectively parallel to ON and OM, we should find that OA and OB contained precisely the same number of units of length as there are units of weight in P and Q. The two sides OA, OB of the parallelogram therefore represent the forces P and Q

acting at O, and the diagonal OR represents their resultant.

The weights P, Q, W may be varied at pleasure, and the result will be found true in every case.

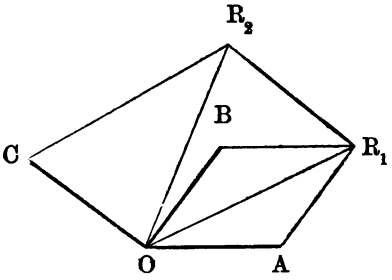
57. From what has been before said, it is plain that we may replace any two forces acting on a point by a single force, which will produce the same effect; and, vice versâ, we may conceive a single force as composed of two forces acting in different directions: and as the same line may be the diagonal of an infinite number of parallelograms, we may consider any force as the result of different pairs of forces, according as is most convenient for illustration or estimation of effects. This is called the *Resolution* of a force in opposition to the term Composition of Forces.

58. It is very convenient, especially in mathematical investigations, to resolve a force into two components at right angles to each other; as, for instance, (fig.) OP may be regarded



as resolved into OM, ON in the directions OX, OY, XOY being a right angle.

59. This composition of two forces is easily carried on by a series of parallelograms to the composition of any number acting on the same point.

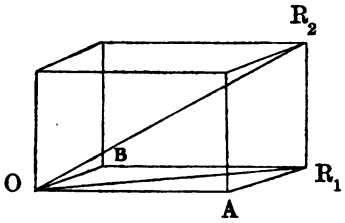


Thus, if OA, OB, OC, (fig.) represent three forces acting on the point O, complete the parallelogram OAR_1B, and OR_1 is the resultant of OA, OB: then complete the parallelogram OR_1R_2C, and OR_2 is the resultant of OR_1 and OC, or of OA, OB, OC, and so on for any number.

This process may also be extended to the case in which the forces lie in different planes, as expressed in the following

60. PROPOSITION. *If three forces, whose directions are in different planes, act together on the same point, and if they are represented in magnitude and direction by the three adjacent edges of*

a parallelopiped, the diagonal of the parallelopiped will represent the resultant of the three both in magnitude and direction.



Thus in (fig.) OR_1 is the resultant of OA, OB, and OR_2, as being the diagonal of the parallelogram OR_1R_2C, is the resultant of OR_1 and OC, or of OA, OB, OC.

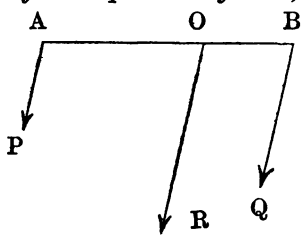
It is evident, that in every case of the composition of force, except when the forces exactly conspire, force is lost.

CHAP. II.

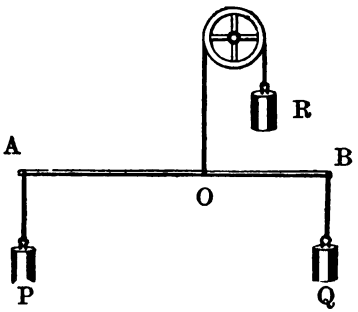
THE COMPOSITION OF PARALLEL FORCES.

61. THE propositions of the last chapter enable us to determine the effect of any number of forces applied to the same point. Forces, however, are very frequently applied to different points of a body or system, and the simplest case is, when the forces are in directions which are parallel to one another. Two *parallel* forces may be compounded according to the following

62. PROPOSITION. *If two parallel forces, as AP, AQ, (fig.) act on the two points A, B respectively, their combined effect will be equivalent to the effect of a single force OR, equal to their sum, parallel to their direction, and applied at a point O which divides the line AB, so that AO is to OB as the force applied at B is to the force applied at A, or AO : OB inversely as the forces.*



This cannot be established strictly without mathematical reasoning, but it may be experimentally exhibited by an arrangement represented in fig. where AB is a strong light rod with two weights P and Q suspended from the ends, and a weight R equal to their sum is attached to a cord



passing over a pully M, and fastened at O. It will be found on trial that the system remains in equilibrio when the point O divides the line AB so that AO : OB : : weight Q : weight P.

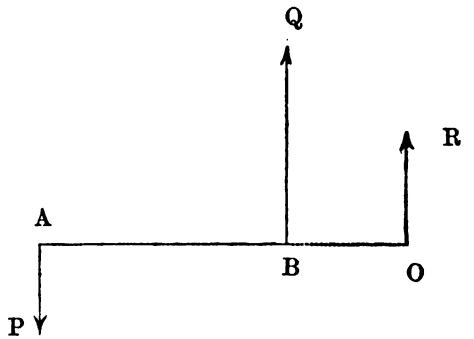
It will be easy to counterpoise the weight of the rod, by hanging on a weight (W) just above or below R, W being exactly equal to the weight of the rod.

From this we may proceed to the composition of any number of parallel forces: in all cases when the forces are in the same direction, the resultant is equal to their sum, and parallel to their direction, but the point of application can only be found by first determining the point of

application for the resultant of two, and then combining this with a third, and so on.

63. If two forces act in directions which are parallel but *contrary*, as in fig. the resultant is found by the following

PROPOSITION. *If two forces, as AP, BQ, (fig.) act on the two points A and B respectively, in di-*



rections which are parallel but contrary to each other, their resultant is a force OR equal to their difference, parallel to their direction, and applied at a point O, in AB or BA produced, according as the force at B is greater or less than the force at A, so that AO : OB : : force at B : force at A.

This may be established by the same experiment as the last Proposition, for we may regard the weight at B as equilibrating the weights at A

and O, or that at A as equilibrating the weights at O and B, and the result will be found to correspond; for evidently, if

$$R = P + Q,$$
$$\therefore Q = R - P;$$

and also if in fig. p. 40.

$$AO : OB :: Q : P,$$
$$\therefore AB : OB :: P + Q : P$$
$$:: R : P.$$

The directions are also all parallel.

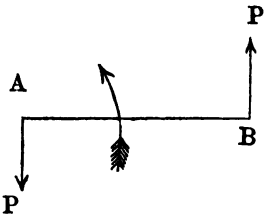
64. There is one very important and remarkable case to which the above Proposition is inapplicable, and that is, when the two parallel and contrary forces are exactly equal. In this case the Proposition would give the resultant equal to zero, and the point of application infinitely distant. In fact, it is evident that no single force could equilibrate such a system, since any reasoning which would give a definite resultant in any position with reference to one force, would give a different one with regard to the other. Such a combination of forces has no tendency to produce any progressive motion in a body or system, but a tendency to rotate or twist round some point intermediate between the direction of the two forces.

Such a system of forces is called a *couple*.

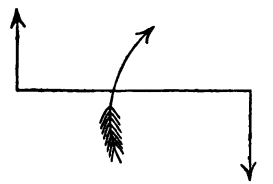
Familiar examples of couples may be remarked in the common operations of turning a corkscrew or a gimblet.

The theory of couples is of great importance in all mechanical investigations. Some writers have deduced every thing else from it. A few of the elementary propositions can only be introduced in this work[a].

65. The *arm* of a couple is the perpendicular distance between their directions, as A B, (fig.) and the product of P multiplied by A B is called the *moment of the couple.*



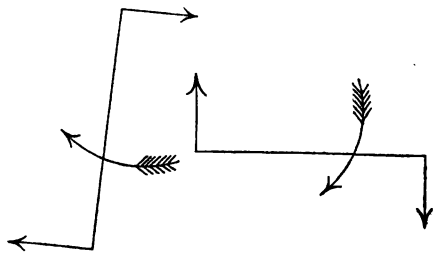
66. Opposite couples are those which would tend to produce rotations in opposite directions. These two figures will represent such.



[a] For the complete development of the theory of Couples, the student is referred to Poinsot's beautiful work, entitled *Elemens de Statique*, or to a pamphlet by Prichard *on the Theory of Couples.*

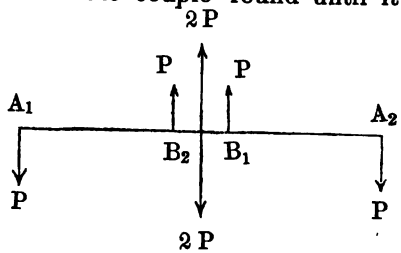
67. PROPOSITION. *A single couple can be turned round in its own plane, without altering its effect upon the system.*

Thus the effect of the couple is the same in either of the positions represented in fig. the forces and the arm being the same.



68. PROPOSITION. *A single couple can be equilibrated by an equal and opposite couple in the same place.*

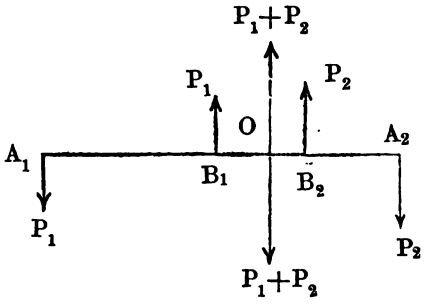
For we may turn the couple round until it comes into the position represented in fig. and then conceiving B_1' B_2 joined as by the dotted line and bisected in O, it is evident that the



effect of the two forces at A_1 and A_2 are by art. 62. equivalent to a single one at O, equal to their sum or 2P; also the other two forces at B_1, B_2 may be replaced by another equal to 2P applied at O in the opposite direction, and this being equal and opposite to the resultant of the other two, will equilibrate the system.

69. PROPOSITION. *A couple will be equilibrated by an opposite couple of the same moment in the same plane.*

For they may be brought into the position represented in fig. and by reasoning as in the last proposition, we conclude that equilibrium will exist, only the point O will not bisect B_1 B_2, but divide it in the inverse ratio of the forces.



70. PROPOSITION. *A couple will be equilibrated by an opposite couple of equal moment acting on the same system. The second couple not being*

in the same plane as the first, but in any parallel plane.

This proposition is self-evident, if we suppose, as we must, that the two couples or the several parts of the system are connected together by rigid inflexible connexions.

Hence it follows, that we may replace a couple,

1. by an equal couple in its own plane; or,

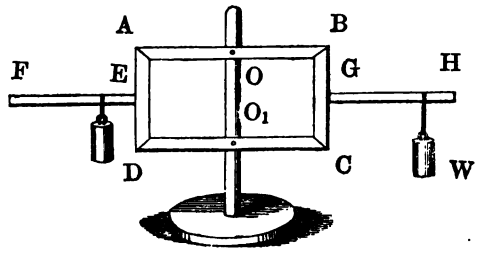
2. by one of equal moment in its own plane; or,

3. by one of equal moment in any plane parallel.

From the above simple propositions we might proceed to other more important conclusions, but these would involve reasoning unsuited to the beginner.

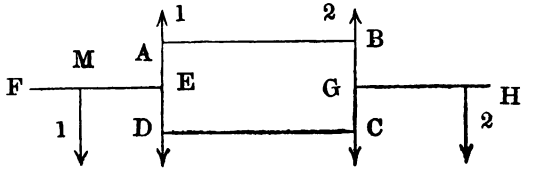
A beautiful illustration of the theory of couples is afforded in the explanation of that which is called Roberval's *Paradox* or Roberval's Balance.

The balance is represented in fig. where A B C D



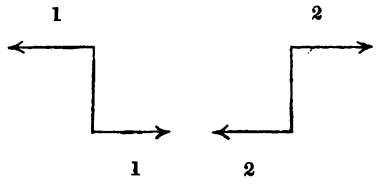
is a *jointed* parallelogram, or a parallelogram with moveable joints or hinges at the angles, and which turns on two points O, O_1, vertically over each other through the middle of the opposite sides AB, CD. To the other sides AD, BC, which always keep in the vertical, two rods EF, GH are attached at right angles to AD, BC, and consequently they are always in the horizontal position, as in the figure. The paradox is, that equal weights as W, W will always be in equilibrio to whatever parts of the arms EF, GH, they are suspended.

This paradox is easily explained by the theory of couples. Suppose four equal forces, each equal to one of the weights W to be introduced into the system, two applied at E in opposite directions, and two at G also opposite, as represented in fig. These forces being equal will not affect the equilibrium. Now the two



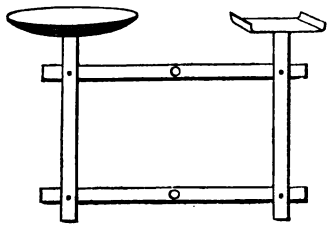
forces marked 1, form a couple whose arm is ME; but these are statically equivalent to another

couple whose arm is AD, and lying in the position indicated in fig. Similarly the couple marked 2, may be replaced by another with arm BC lying as in fig. The forces which constitute these two couples are obviously neutralized by the reaction of the pivots, and there now remain only the two forces at E and G, which are plainly in equilibrio.



The apparently unequal leverage of the two weights, is thus seen to affect only the strain on the points.

A weighing machine, constructed on the principle of this paradox, is now in general use. The only difference is, that instead of arms attached to the middles of the vertical sides of the parallelogram, the pans are attached to



them at the top, as exhibited in fig. The convenience of this balance is obvious.

72. To return to the consideration of parallel forces in general. It is obvious, in the simplest case of two parallel forces, as represented in fig. p. 39, that if the two forces P and Q take different directions, still continuing parallel, and applied to the same points and with the same magnitude, the position of the point O will not be changed, the resultant will merely take a new direction parallel to the new direction of the forces. The same will be true if there be any number of forces in the system which continue of the same intensity, and applied to the same points; the resultant of all will be applied to a point which will remain invariable, although the forces take new positions, remaining parallel to each other.

This invariable point, through which the resultant of the system passes, is called the *centre of the Parallel forces.*

CHAP. III.

ON THE CENTRE OF GRAVITY.

73. WHEN describing the general properties of bodies, we remarked (art. 27.) that all terrestial bodies had a tendency towards the earth. This tendency or force varies at different parts of the earth's surface, but at one and the same place, or with reference to the molecules of the same body, it may be regarded as constant. The direction of this force or tendency is strictly towards the centre of earth, but for the same body this may be taken as perpendicular to the horizontal plane. Its direction is easily ascertained by means of a plumb line, or a perpendicular to a sheet of still water.

Every molecule of a body is thus solicited by a force drawing it down, and the directions of the forces on the several molecules are all parallel to one another. This will also be the case if the body be turned over or made to take different positions: in every position it is acted on by a number of parallel forces. By art. 72. these parallel forces have a resultant which will in all positions of the body pass through an invariable point, which we

there called the centre of parallel forces. When speaking of gravity, this point is called *the centre of gravity.*

74. The *weight of a body* may be regarded as the resultant of all the forces acting on the several molecules, or the resultant of the weight of the molecules.

75. The determination of the position of the centre of gravity is a point of vast consequence in all circumstances of rest or motion. In some bodies it is readily ascertained; in some, mathematical formulæ enable us to find it; and where these fail, it often happens that we can fix its place by actual experiment.

76. All symmetrical bodies which are homogeneous have their centre of gravity coincident with the centre of magnitude.

Thus, the centre of gravity of a homogeneous straight line or thin prismatic or cylindrical stick is in the point of bisection.

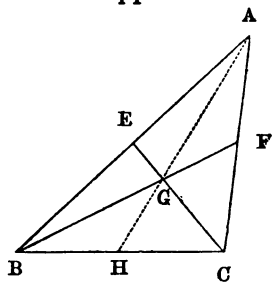
The centre of gravity of a parallelogram is in the intersection of its two diagonals.

The centre of gravity of circles, ellipses, and so of spheres, spheroids, &c. is in the geometrical centre.

77. The centre of gravity of a triangle lies in the

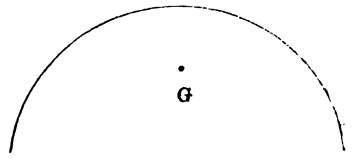
intersection of two straight lines drawn from any two angles to the bisection of the opposite sides.

For instance, (fig). if BF, CE be drawn from the angles B, C to the points F, E, which are the points of bisection of the sides AC, AB, the centre of gravity is at G, where these two lines intersect; and it is easily shewn, that $BG = \frac{2}{3} BF$, $CG = \frac{2}{3} CE$, and if AH be drawn, $AG = \frac{2}{3} AH$.



78. The centre of gravity of any polygonal figure may hence be determined, since it can be divided into triangles, and the centre of gravity of each triangle being found, we may proceed as in the composition of several parallel forces.

79. The centre of gravity of a body is not always within the body itself. The centre of gravity of a circular hoop is in the centre of figure, and the centre of gravity of the arc of a circle is at an imaginary point, as at G. (fig.)



80. PROPOSITION. *If the centre of gravity of a body is supported, the body will rest in all positions, unless some other force act upon it.*

This is self-evident, since the resultant of all the forces thus passes through a fixed point.

81. This proposition leads to a practical definition of the centre of gravity, which is of much use in enabling us to determine the centre of gravity of a body by experiment; for the centre of gravity may then be said to be the point about which a body or system rests in all positions.

Hence to find the centre of gravity of a body by experiment.

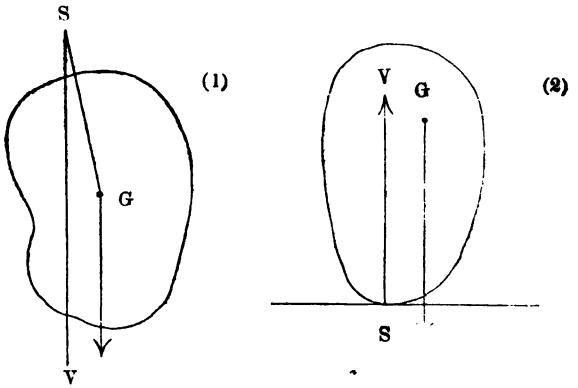
Suspend the body by a string, the body will not remain at rest until the centre of gravity is vertically under the point of suspension, for in any other position the weight would not be sustained. The direction of the string thus gives a definite line in which the centre of gravity lies, and if this can be marked, we have only to suspend the body by a different point, and we obtain another line which passes through the centre of gravity, and the intersection of these two lines is the point required.

Or, we may balance the body upon a rod or bar, and we know that the centre of gravity is in the vertical plane which contains the bar, and

by taking two other positions, we get three planes which contain the centre of gravity, which must therefore be in the point in which the three planes intersect.

82. PROPOSITION. *If a body be supported by a single point, the centre of gravity must always be in the vertical which passes through that point.*

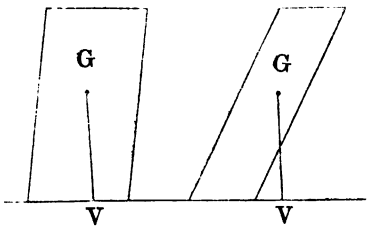
In no other position could the weight of the body be counteracted, whether the point of support be a point of sustension, or a point upon a hard plane. This is evident by inspection of figures; thus, in fig. 1. which is intended to re-



present a body hung by a string to a point S, unless G the centre of gravity be in the vertical line S V, the effect of the weight is not sustained. So in fig. 2. where the point of support is a hard

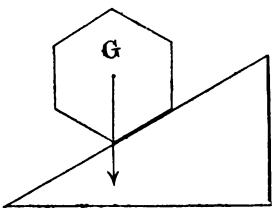
plane, the plane will only counteract a force opposed to SV, the vertical through S, the point of support; unless therefore G the centre of gravity be in that vertical, the body will roll over.

83. PROPOSITION. *If a body is supported on any base, the vertical through the centre of gravity must fall within the base.*



The figures will illustrate this proposition: in the first the effect of the weight is counteracted; in the other it is not, and the body will fall over.

This proposition applies when the plane of support is inclined to the horizon as well as when it is horizontal, as may be seen in fig. but in this instance, unless friction or some impediment interfere, the body will *slide* down the plane.



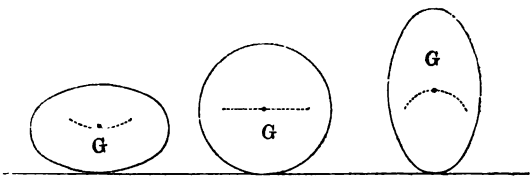
84. In these cases there are different kinds of equilibrium. The equilibrium may be *stable, neutral,* or *unstable.*

Stable equilibrium is that in which, if a slight disturbance take place, the body tends to return to its former position.

Neutral equilibrium is that in which, if the body be disturbed, there is no tendency to return to or to go further from its original position.

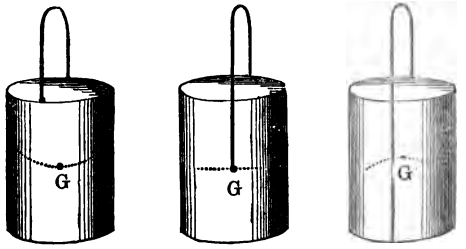
Unstable, is that in which the slightest disturbance will cause it to move further from its position.

In the case of bodies resting on a point upon a horizontal plane, the three kinds are exhibited, (1) by a spheroid, or an egg lying on its side; (2) a sphere or globe; (3) a spheroid on a smaller end. See figs.



In the first of these, a slight disturbance causes the centre of gravity to move higher: in the second, it does not affect the distance of G from the plane; in the third, it sends it lower.

In the case of bodies hung up by a point, the same effects are at once seen by referring to a common bucket, as in figs.



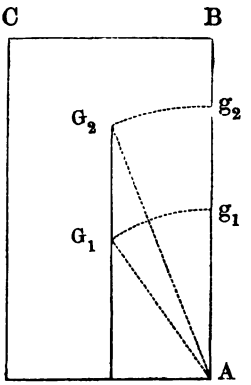
When suspended by a handle in the ordinary way, we have *stable* equilibrium: if the handle were attached to points, such that the line joining them passed through the centre of gravity, the equilibrium would be *neutral:* and if attached below G, it would be *unstable.*

From these remarks the following propositions are easily deduced.

85. PROPOSITION. *The centre of gravity of a body always takes the lowest position it can, compatible with the conditions which affect the body.*

For if not, it must descend.

86. PROPOSITION. *The lower the centre of gravity, the more stable the equilibrium, ceteris paribus.*



This will be understood by reference to fig. in which ABCD may be supposed to represent a section of a body with AD as a section of its base, if the centre of gravity be at G, it cannot be overset in the direction indicated by the arcs, except by making G, describe the arc G_1g_1 which subtends at A the angle G_1Ag_1, but if the centre of gravity be higher, as at G_2, the centre of gravity will only have to move through the arc G_2g_2, which subtends at A the smaller angle G_2Ag_2.

The application of the principles laid down above are of every day importance, and admit of many illustrations.

87. It follows from the definition of the centre of gravity, that when we wish to estimate the effect of the weight of any body or system of bodies, we may regard it as a single equivalent force applied at that point.

CHAP IV.

ON SIMPLE MACHINES IN EQUILIBRIUM.

88. A machine is an instrument or contrivance for transmitting or modifying certain powers at our disposal, so as to produce certain effects. It is used either to economize power, or to produce a greater velocity.

Machines have to be considered both statically and dynamically; but we investigate first the conditions of equilibrium, and then it is easier to pass on to the circumstances and effects of motion.

89. In all machines there are three things to be considered.

1. The force or power which we are able to apply, and this is technically called the *Power*, whatever its nature.

2. The resistance to be overcome, or the work to be done, and this is commonly called the *Weight* or *Load*.

3. The machine or apparatus itself.

Both Power and Weight are frequently expressed by weights, and these give us practically a means of comparison. If the power be that of steam, we may speak of it as producing a pressure of so many pounds on the square inch, &c.

90. In the exposition of the theory of machinery, it is necessary to omit, at first, many circumstances, which must however be accurately estimated, before we can apply our results for any practical purpose. For instance, we begin by omitting all the effects of friction, the surfaces are assumed to be perfectly smooth and polished; the solid parts are supposed to be absolutely rigid and inflexible. If ropes are employed, we omit the effects of their stiffness, and suppose them flexible, inextensible, and without weight. Again, at the first we omit all consideration of the weight and inertia of the machine, the resistance of the air, &c. And this method of proceeding is the most philosophical. Our first conclusions are indeed only first approximations to the truth, but we can afterwards take into account one by one the various considerations we had omitted, and so determine the precise effects of each circumstance separately, and approximate more close to the true conclusion.

91. It may be observed at once, that in machinery of whatever kind, power is not *created* but *converted:* and although the statement of the effect of any machine is sometimes made so as to sound like a paradox, yet in no case can a small power produce a great effect without in some way or other employing an effort equivalent to the result. A force of 1lb. may indeed by the intervention of machinery move a weight of 10,000lbs. but it will be by exerting itself through a proportionably large space.

If, for instance, the large weight of 10,000lbs. be carried through 1 foot, the 1lb. force will have to move through 10,000 feet. We shall put this in a more general form presently.

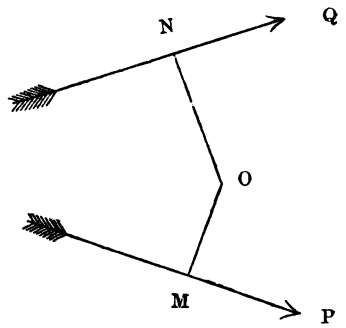
92. In all machines, there are certain fixed *points, props,* or *fulcra,* which are so arranged that the portion of the weight which does not immediately act against the power is distributed amongst them. Sometimes these fulcra are fixed *lines* or *surfaces* capable of resisting pressure.

93. It may be here remarked, that fixed *points* are capable of resisting pressure in all directions, fixed *lines* and *surfaces* only in a direction at right angles to them. The resistance of a fixed

line is therefore limited to the plane to which the line is perpendicular, the resistance of a surface is limited to the single line which is perpendicular to the surface.

The following definitions will be of great use in estimating the efficacy of the different mechanical powers, and of machinery in general.

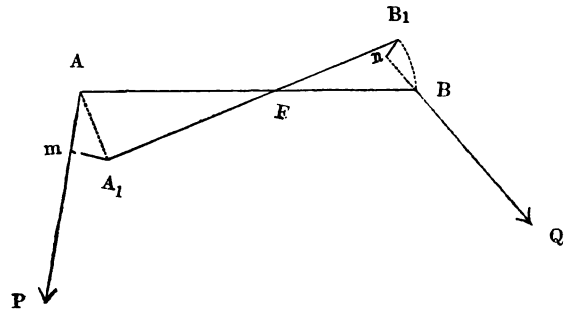
94. *The moment of a force with respect to a given point* is the product of the force and the perpendicular let fall on its direction from the given point. Thus if (fig.) P and Q are any two forces whose directions are indicated by the arrows, and O any given point; and if from O perpendiculars OM, ON are let fall on their directions,



$P \times OM$ is called the moment of P with reference to the given point O, and $Q \times ON$ is the moment of Q with reference to the same point. These moments are also said to be *positive* or *negative*, according as their effects with regard to O are in

the same or opposite directions. In the figure, the two forces have opposite moments with reference to O: whereas if the point were above or below both forces, they would have the same sign.

95. The *virtual velocity* of any force of a system is the very small space through which the point to which the force is applied would "in ipso motus initio," move in the direction of the force, if a motion is communicated to the system. The motion of the point may be altogether in the direction of the force, but it will not generally be so on account of the connexion of the parts of the system. This definition may be illustrated by fig. where AFB represents a bar moveable



in the plane of the paper round a point F, so that when a motion is given, every point in the bar describes an arc of a different circle, though each

arc subtends the same angle at F. Suppose forces P and Q to be acting on this in the directions AP, BQ, and then a motion to be given to the bar which causes it to take the position A_1FB_1, the point A has really described the arc AA_1, but the motion of A in the direction of P is represented by Am, the angle AmA_1 being a right angle; similarly the motion of B in the direction of Q is exhibited by Bn, if BnB_1 is a right angle; and if the motion given to the bar be very small, Am, Bn represent the *virtual velocities* of the forces P and Q. Other illustrations will occur as we proceed.

These definitions being premised, the following Propositions will express the condition of equilibrium in machines.

96. PROPOSITION. *When a machine is in equilibrium on a single fulcrum, the moments of the power and weight with reference to that fulcrum are equal and opposite.*

97. PROPOSITION. *In every machine in equilibrium, the product of the power and its virtual velocity is equal and opposite to the product of the weight and its virtual velocity.*

These two propositions may be extended to

cases where there are any number of forces or weights; and if we express the moments in one direction by positive or + signs, and those in the opposite by negative or — signs, we may enunciate the Proposition of art. 96, in the following general

98. PROPOSITION. *When a machine is in equilibrium on a single fulcrum by the action of any number of forces, the* sum *of the moments of all the forces with regard to the fulcrum is equal to zero.*

By sum is meant the Algebraical sum, each product having its proper sign.

In the same manner, by regarding the virtual velocities in one direction as positive, while those in the other are negative, we may enunciate the Proposition of art. 97, thus:

99. PROPOSITION. *When a machine is in equilibrium, the* sum *of the products of the forces and their virtual velocities is equal to zero.*

This Proposition is indeed only a particular case of the great principle of virtual velocities. This principle is made by Lagrange the basis of his great work, the Mecanique Analytique. It is of great importance and universal application;

for a demonstration, the mathematical student is referred to the work of Lagrange, or to the many mathematical works on Mechanics.

100. From the two last Propositions, we at once arrive at that which is the golden rule in all machinery, viz. that *what is gained in power is lost in velocity.* The truth of the rule is implied in the Propositions referred to, as will be evident by exhibiting the proposition of art. 97, thus:

P × P's virtual vel. = W × W's virtual vel.

∴ P : W :: W's virtual vel. : P's virtual vel.

We now now proceed to apply these Propositions to the simple machines.

101. It is usual to consider the simple machines, or, as they are sometimes called, the mechanical powers, as six in number, viz. the *lever*, the *wheel and axle*, the *inclined plane*, the *wedge*, and the *screw*.

102. But these six may be reduced to the following three classes.

1. Those in which there is employed a solid body moveable round a fixed axis, and this class includes the lever, and wheel and axle.

2. Those in which force is transmitted by cords or bands, and this includes all the various kinds of pulleys.

3. Those in which there is employed a hard surface inclined to the horizon, and this includes the inclined place, the wedge, and the screw.

But we will consider each of the six mechanical powers separately.

SECTION I.

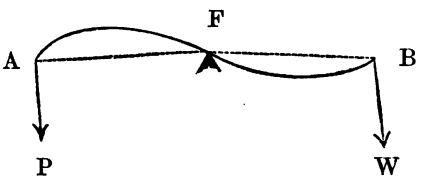
THE LEVER.

103. A lever is any inflexible or solid rod turning on a fixed axis.

The fixed axis is called the fulcrum, and the lever is at first supposed to be without weight.

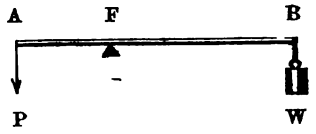
The lever may be straight or bent. The *arms* of the lever are the distances of the points of application of the forces from the fulcrum. In the crooked or curved lever, the *arms* are measured in the straight line.

Thus in the curved lever AFB, the arms are the distances represented by the dotted lines.

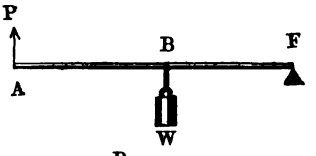


104. There are three kinds of levers.

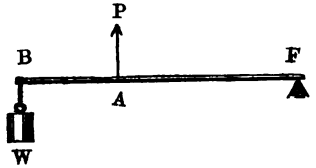
First, when the fulcrum is between the power and weight. (fig.)



Second, when the weight is between the fulcrum and power. (fig.)



Third, when the power is between the weight and the fulcrum. (fig.)



105. In each of these forms the condition of equilibrium is the same; and if the directions of the power and weight be parallel, as in the above figures, we obtain from art. 96,

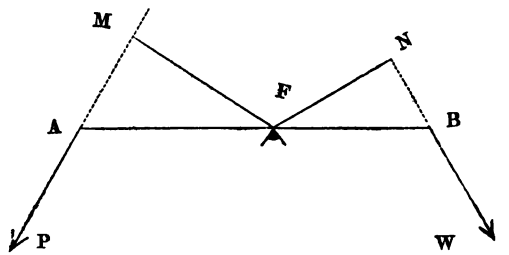
$$P : W :: FB : FA,$$

or the power is to the weight inversely as the arms. The arms are in this case the perpendiculars let fall in the direction of the forces.

When either power or weight is oblique to the lever, we have to substitute the perpendiculars let fall on their directions for the arms: thus in

fig. if FM, FN are perpendiculars let fall from F on the directions of P and W,

$$P : W :: FN : FM$$
$$:: FB \sin FBW : FA \sin FAP.$$



The same formulæ will apply to bent levers as well as to straight.

107. From the formula of equilibrium it follows that we increase the effect or leverage of the power, either by increasing the arm on which it acts, or diminishing the arm at which W acts. There is, however, a limit in practice, for too long a lever is heavy, and liable to break.

108. In all cases, a load is thrown on the fulcrum, which must be estimated.

In levers of the first order, the pressure on the fulcrum $= P + W$.

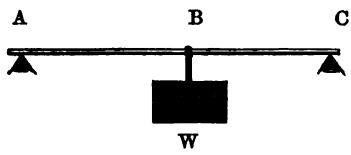
In those of the second and third, it $= P - W$.

If several weights or forces act, the Proposi-

tions in arts. 98, 99, at once give the formula of equilibrium.

109. The weight of the lever itself may easily be taken into account, by regarding it as collected in its centre of gravity. It is then only another force applied at a given point.

110. When a bar rests on two props, and supports at some intermediate point a given weight, the pressure on the props is at once determined by the principle of the lever.



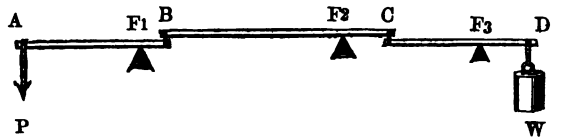
Thus, if ABC (fig.) represent such a bar,

the pressure on prop at A : pressure on prop at C :: BC : BA,

or inversely as the distances of the point from which W is suspended.

111. The condition of equilibrium in a *compound* lever is obtained, by combining the effect of each one separately.

Thus if (fig.) AF_1B, $BF\ C$, CF_3D represent a system of three levers, in which the end B of the



first acts up on the second, and the end C of the second acts down on the third, and F_1, F_2, F_3, are the fulcra, we have,

$$\begin{aligned} P &: \text{Force at B} :: BF_1 : F_1A. \\ \text{Force at B} &: \text{Force at C} :: CF_2 : F_2B. \\ \text{Force at C} &: W :: DF_3 : F_3C. \end{aligned}$$

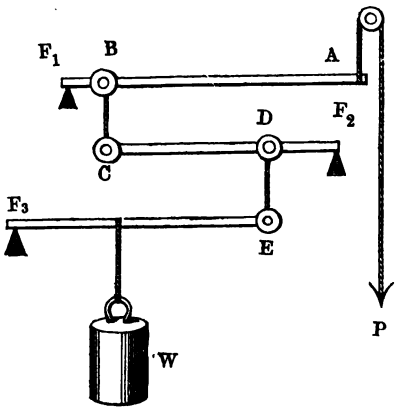
Whence by composition,

$$P : W :: BF_1 \times CF_2 \times DF_3 : F_1A \times F_2B \times F_3C.$$

Or we may at once apply the principle of virtual velocities, and then we have,

P : W :: W's virtual vel. : P's virtual vel.

112. Levers of the second order are frequently arranged as in fig. Where AF_1, CF_2 are connected by the *link* BC, and CF_2, EF_3 are connected by the *link* DE.



113. The applications of the lever are very numerous, and many familiar examples will

occur to the student: the following statement will exhibit the various purposes to which a lever may be applied:

1. With a small power to hold in equilibrio or set in motion a great weight; e. g. a *crow bar*, &c.

2. To change the direction in which forces are acting. For this purpose a bent lever is frequently employed, as in the common *bell pulls*, &c.

3. To communicate motion from one point to another, and either to produce the same velocity, or a velocity bearing a certain ratio to the velocity at command, or sometimes to convert a reciprocatory motion into a rotatory. The lever when thus employed is called a *rod* or *link*. The foot board of a turning lathe and the crank are familiar examples of *link-work*.

4. To magnify motions so minute as not otherwise to be perceptible. This is accomplished by means of a lever of unequal arms, and sometimes by a lever of the first, sometimes of the third order: e. g. in the *pyrometer*, *wheelbarometer*, &c.

5. To estimate the amount of pressure, or the mass of a body. This is a most important application of the lever. There are various kinds of balances, but the *common balance* and *the steel-yard* may be here described.

THE COMMON BALANCE.

114. The common balance or *scales* is, in its usual form, a lever of the first order. It consists of a beam (fig.) supported at its middle point with pans or dishes suspended from its extremities. In one pan is placed the substance to be *weighed*, and known weights in the other.

115. The requisites of a good balance are,

1. The *horizontality* of the beam when equal weights are placed in the pans.

2. The *sensibility* of the system, that it may indicate any slight difference in the weights; and the sensibility is measured by the angle through which the beam will be deviated by a known small weight.

3. The *stability*, that it may tend to return to

the position of equilibrium if any disturbance take place.

116. The first of these requisites is secured by the symmetrical construction of the beams, pans, &c. and by thus taking care that the centre of gravity of the whole may be in the line which is perpendicular to the beam through the point of suspension.

117. The second is secured in practice principally by excellence of workmanship, which diminishes friction. The beam is fitted at the point of suspension with *knife edges* of very hard steel; these in the common balance rest in a loop of metal, and the deviations of the beam are observed by an index which vibrates within the sides of the loop. In philosophical balances, or those used in weighing precious stones, &c. the knife edges rest on agate plates polished very truly: the ends of the beam are also furnished with knife edges, on which are laid agate plates, to which the pans are attached by silk cords.

It is shewn by simple mathematical investigation, that, *ceteris paribus*, the sensibility of a balance is increased,

α) By lengthening the arms.

β) By diminishing the weight of the beam.

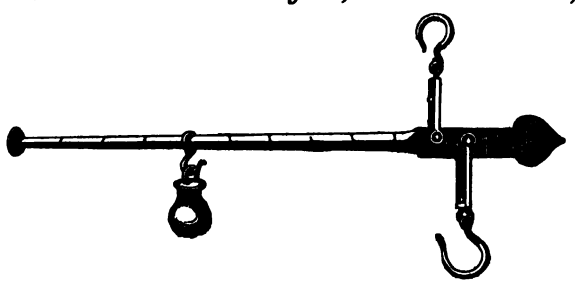
γ) By shortening the threads by which the pans are suspended, or, in other words, by bringing the centre of gravity near the point of suspension.

δ) By diminishing the load; or the balance is more sensible when used for small weights.

118. The third requisite of a good balance depends on the centre of gravity being below the point of suspension, and the lower it is, the more stable, as may be inferred from art. 86.

Hence if we increase the stability, we diminish the sensibility. (art. 117. γ.) The best way of combining the requisites, is to lengthen the arms (art. 117. α.) without increasing the weight, (art. 117. β.) but this may be at the expense of their rigidity or strength.

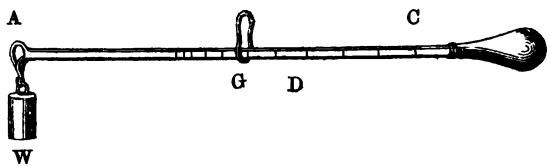
119. The *common steelyard*, or Roman balance,



is a lever of the first order, consisting of a

beam supported by knife edges in a loop, and having one arm longer than the other. The body to be weighed is suspended at the end of the shorter arm, and its weight is ascertained by sliding a given weight along the other arm. The larger arm is graduated, and the weight of the body is read off on the graduation. The divisions of the arm are equidistant, or form a series of arithmetical progressionals measuring from the fulcrum.

120. In the Danish steelyard, (fig.) the counterpoise is fixed at one end, but the fulcrum is moveable. The divisions of this are unequal,



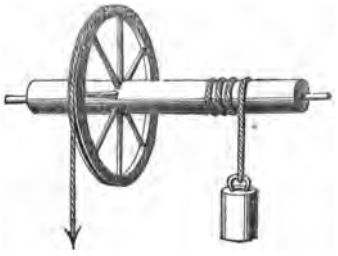
and if the distances are measured from the end to which the body to be weighed is suspended, they form a series of harmonical progressionals.

SECTION II.

THE WHEEL AND AXLE.

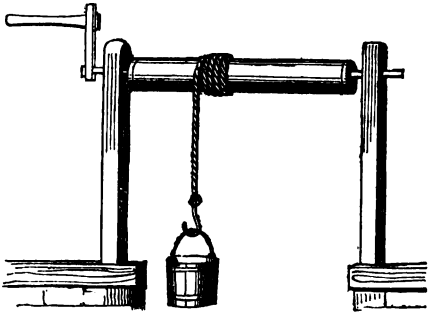
121. The *wheel and axle* (fig.) consists of a cylinder which is supported on pivots at its

extremities, and is capable of revolving about those pivots; and to this cylinder is attached a wheel of larger diameter, with its centre in the axis of the cylinder, and so that the two revolve with one common motion.



The weight is suspended by a cord which winds round the cylinder or axle, the power is applied to the circumference of the wheel.

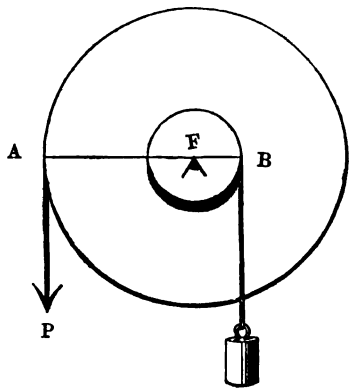
122. The power may be that of man, or an



animal, or inanimate matter, and the way of applying the power is various. Sometimes the

wheel is furnished with projecting spokes, to which the human hand may be applied; sometimes a single crooked arm, or winch, is employed instead of the wheel, as in fig. Sometimes an animal is employed within a large wheel or barrel; sometimes a weight is suspended by a string, and coiled round the wheel, as in fig. p. 77; or the force or weight of water is applied by means of floats or buckets, as in water-wheels.

123. The condition of equilibrium is easily derived from the lever; for as we assume the construction perfectly rigid, we may conceive the power and weight to act in the same plane, and then we have the case represented in fig. whence,



$$P : W :: FB : FA$$

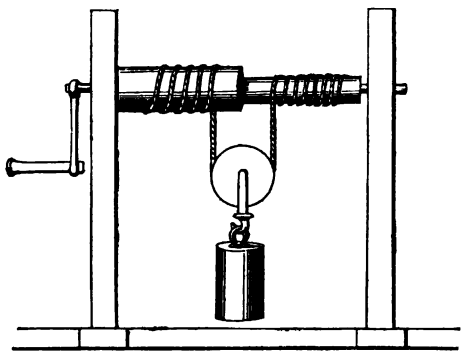
:: radius of axle : radius of wheel.

This formula at once follows from the principle of virtual velocities; for here, if motion be

communicated, the actual velocities are proportional to the virtual velocities; and if one turn be given, we have,

vel. of P : vel. of W : : circumf. of wheel : circumf. of axle.
: : radius of wheel : radius of axle.

124. We may increase the efficacy of a given power either by increasing the radius of the wheel, or diminishing the radius of the axle; but in the one case we make the size inconvenient, in the other the axle becomes weak. The Chinese wheel and axle obviates these disadvantages, by employing an axle of two parts, as in fig. one



part thicker than the other, the rope is made to pass through a fixed pulley, to which the *weight* is attached, and to roll off the smaller

part of the axle while it rolls on the thicker. The condition of equilibrium here is,

P : W :: ½ difference of radii of parts of axle : rad. of wheel.

The efficacy is unlimited, since we may diminish without limit the difference of the radii of the thicker and the thinner parts of the axle.

In this proportion, the *half* difference of the radii is taken, and not simply the difference as might at first be expected. The reason of this may be seen by applying the principle of virtual velocities. Let one turn be given to the machine, the power, as in the simple wheel and axle, moves through a space equal to the circumference of the wheel, the part of the rope which is not coiled on either part of the axis is shortened by a length equal to the difference of the two circumferences; but since this rope passes under the pulley, each side of it is only shortened by half this difference, and therefore the weight is raised only by this amount.

125. *Compound wheels* and axles are exactly analogous to the compound lever, and they are employed to communicate a force or motion through a greater space, or to produce different velocities. The power is supposed to act on the circumference of the first wheel, whence its effect

is transmitted to the first axle; the circumference of this first axle acts on the circumference of the second wheel, whence it is transmitted to the second axle, which in its turn acts on the circumference of the third wheel, and so on. The formula of equilibrium is,

P : W : : product of radii of axles : product of radii of wheels.

There is in fact nothing different in the mechanical effect of such a combination from that of a system of levers, except that it admits of a continuous action, and produces continued and regular motion.

126. There are different methods of causing the circumferences to act on one another:

α. Sometimes the friction of surfaces is employed, as when the axles are simply pressed against the wheels. This is called *Rolling Contact*, and in this case the friction is increased by cutting the wood of which they are made across the grain, or by facing them with rough leather, or by chalking them.

β. Sometimes by bands, or belts, or straps. This is called *Wrapping Connection*, and the bands may be crossed or no according to the direction required. This method properly belongs to the next class of machines, in which a

power or a motion is transmitted by cords or ropes, but as the bands or belts generally pass over wheels or axles of different radii in order to communicate different velocities, the efficacy must be estimated by the radii as above.

γ. Or most frequently by teeth or cogs upon the circumference of the wheels and axles, the teeth working one into another. This is called sometimes *Wheel and Pinion* work, the axles being called *pinions*, and their teeth *leaves*.

127. There are three kinds of toothed wheels distinguished by the position which the teeth bear with respect to the plane of the wheel. When the teeth are in the plane of the wheel, the wheel is called a *spur* wheel. When they are at right angles to the plane of the wheel, it is a *crown* or *face* wheel, and such a wheel is employed to transmit motion to another at right angles to it. If the teeth are oblique to the plane of the wheel, the wheel is a bevilled wheel; and such wheels are used to produce motion round an axis either at right angles, or any other angle, according to the inclination of the teeth.

128. In toothed wheel work, the teeth and leaves must be made similar, and to fit into one another, so that the number of teeth is a measure

of the circumference and proportional to the radius; hence the efficacy of such combinations may be readily found by counting the teeth, and we have,

$$P : W :: \left\{\begin{array}{c}\text{product of teeth}\\ \text{of all the pinions}\end{array}\right\} : \left\{\begin{array}{c}\text{product of teeth}\\ \text{of all the wheels}\end{array}\right\}$$

129. The shape of the teeth is a question of great importance; if they were merely pegs or pins, there would be a scraping and a jolting motion, and hence great wear and tear. In the construction of teeth the following points should be attended to:

α. The teeth of one wheel should always press at right angles to the radius of the other, or else force would be lost.

β. As many teeth as possible should be in contact at the same time, that one may not have to resist the whole strain.

γ. The direction of pressure should be the same during the whole action, in order to ensure uniformity of effect.

δ. The surfaces must roll over one another, and not rub or scrape.

130. These conditions are fulfilled by constructing the teeth in the form of part of the curve known as the involute of the circle, or

else in that of an epicycloid. These two curves are commonly employed, but various other forms will satisfy the conditions. On this subject and on the Elements of Machinery in general, the student should consult the master work of Professor Willis, entitled, "the Principles of Mechanism."

SECTION III.

THE PULLEY.

131. *The pulley* consists of a wheel called a *sheave*, fixed in a block, and turning on pivots; round its rim is a groove over which a cord may pass. The pulley may be either *fixed* or *moveable*.

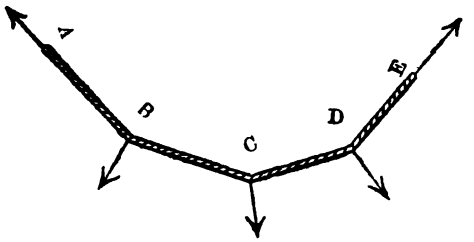
Although the term pulley implies the combination of a rope and a wheel, yet the effect of the machine depends altogether on the rope; the wheel is introduced solely for the purpose of the diminishing the effects of *friction* and *rigidity*, or the imperfect flexibility of the rope. The consideration of both these is omitted in our first approximation, and hence our formulæ of equilibrium may be derived without regarding the wheel, or that which is more properly called the pulley. As soon, however, as motion is given, the wheel is most important, for it transfers the friction which would arise from the roughness of

the rope to the polished pivot, and by turning the rope over a curve instead of at a sharp angle it gets rid of the effect of imperfect flexibility.

132. In deducing then the formulæ of equilibrium for the different combinations and arrangements of pulleys, we consider whether the force is transmitted by the same cord, or by different ropes or *cords* acting on one another.

If a cord is perfectly flexible, (as theory first supposes it to be,) it is capable of resisting only those forces which are applied in the direction of its length. Hence in the simple case of a cord tied to a fixed point, if a force be applied to the other end, the cord will not rest until it takes along its whole length the direction of the force. The cord is assumed to be without weight, or else we have another force acting. In the case supposed, the cord when at rest transmits along its whole length the force applied to the end, and this force is neutralized by the reaction of the point. Every part of the cord is pulled in equal and opposite directions by a force equal to the force at the end, and this force is called the *tension* of the cord. Hence we may state, as the principle of the action of cords, that when a cord is at rest by the action of any number of forces, and in consequence takes a polygonal form,

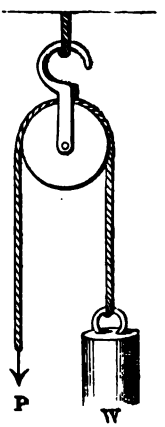
as ABCDE in fig. the part AB undergoes equal and opposite tensions in the direction of

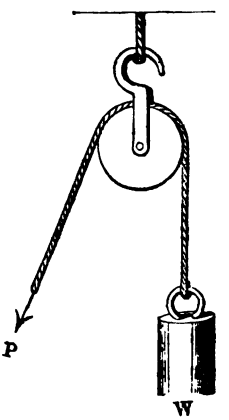


its length; thus also any part CD undergoes equal and opposite tensions in the direction of its length, &c.

From this principle, the formulæ of equilibrium in the different kinds of pulleys is at once obtained.

133. The simplest case is that of the *single fixed pulley*, represented in figs. Here there is one and the same rope, which has the same



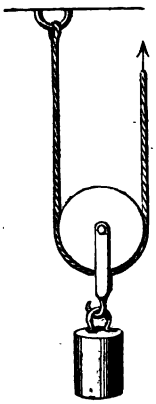


tension in every part, in equilibrium therefore

$$P = W.$$

Such a contrivance is very useful for changing the direction of a force, or for conveniently applying a force. Several fixed pulleys are used to transmit a force in different planes.

134. The *single moveable pulley* introduces the resistance of the point of support, as fig. Here the cord has the same tension in every part, and the tensions of the two parts support the weight,



$$\therefore P = \frac{W}{2}$$

135. This supposes the ropes parallel, but if they are inclined at an angle, (as in fig. 1. overleaf,) the two tensions are pulling partly against one another, and the formula must be determined by resolving the tensions, and ascertaining how much is really effective in supporting the weight. This may be done (as in fig. 2. overleaf) where the pulley is omitted and the weight supposed to be immediately attached to the string. If CP and CQ, each equal to one another, represent the tensions of the parts of the string, they produce by the parallelogram of

forces an equivalent represented by the diagonal CW which therefore represents the weight sustained, and CW = 2CM, whence by *Trigonometry*

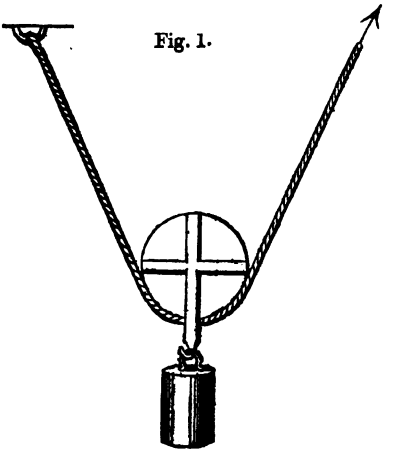


Fig. 1.

$$P = \frac{W}{2 \cos MCP} = \frac{W}{\cos \frac{\theta}{2}}$$

if θ = angle made by the parts of the string.

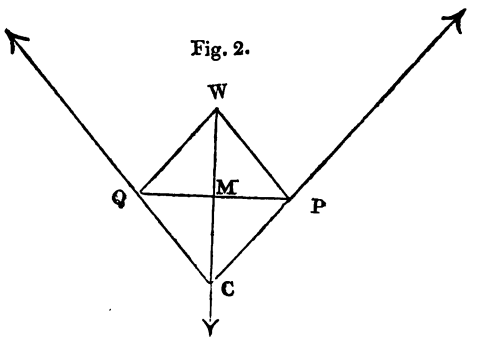
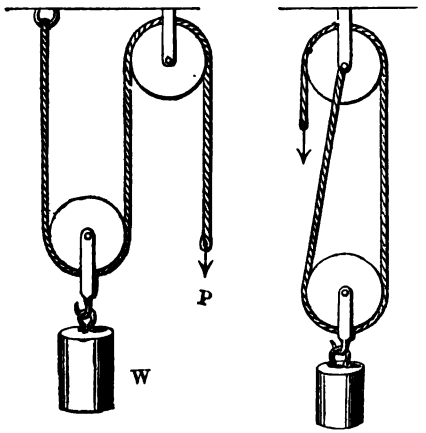


Fig. 2.

In the figure, MP, MQ represent the parts of the tensions *lost*, by pulling against each other.

136. In practice, the single moveable is

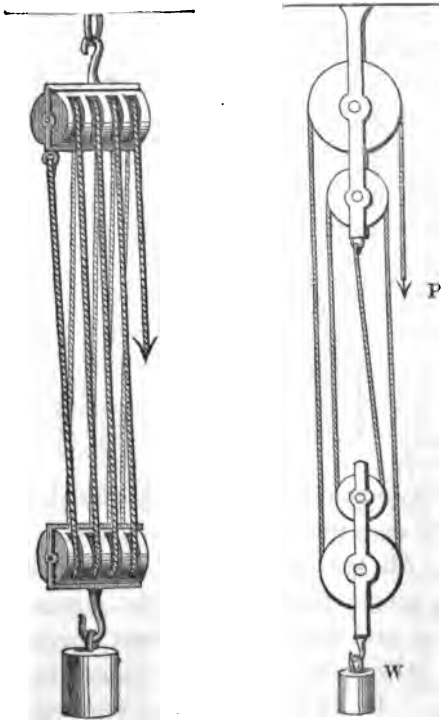


generally combined with a single fixed, as in figs. but this does not change the efficacy.

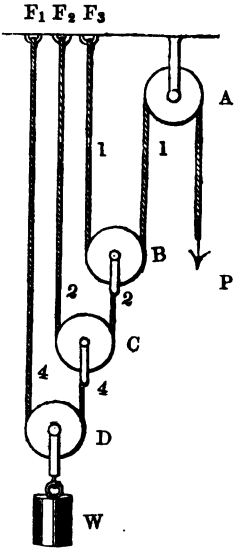
137. All systems of pulleys are combinations of the single fixed and single moveable: the simplest and most usual system is represented in fig. where the same rope passes over every sheave. In these cases the tensions of the several parts of the rope combine to support the

weight, and if n = the number of strings at the lower block,

$$P = \frac{W}{n};$$



the parts of the rope being supposed parallel.



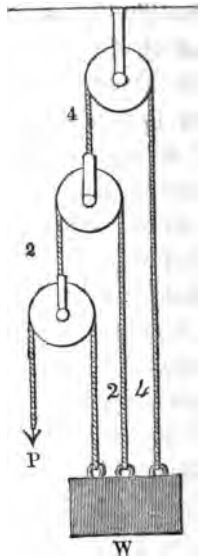
138. The system represented in fig. is generally described as the one in which each pulley hangs by a separate string Its efficacy may be thus seen; the same string passes over A under B, and is attached to F_1, the two parts numbered 1 in fig. combine to hold up pulley B, and each exert on B a force equal to P, hence pulley B and the string attached to it is sustained by a force 2P. This force is transmitted along its whole length, and hence under C up to F_2. The pulley C is sustained by the two parts numbered 2 in fig. and as the tension of each of these is 2 P, the whole force on C is 4 P. The two parts of the string which pass under D hence undergo a tension of 4 P, and the whole force on D is 8 P: and so on, each additional pulley doubling the effect; hence with three moveable pulleys as in fig.

$$W = 8\,P = 2^3P.$$

and generally with n moveables,

$$W = 2^nP.$$

139. The system represented in fig. is that in which each string is attached to the weight. The formula is easily ascertained according to the principles laid down but here the sum of the forces or tensions of the strings, are equal to the weight, and



$$W = \text{tension of string } 1 + \text{tension of string } 2 + \text{tension of string } 3 + \&c.$$
$$= P + 2P + 4P + \&c.$$
$$= P(1 + 2 + 4 + \&c.)$$

and if n = number of strings,

$$W = 2^{n-1} P$$

140. In the above we have neglected the weights of the pulleys. These however are easily added in.

In the cases represented in figs. p. 87, 90, the weight of the lower block may be taken as part of the weight supported. In fig. p. 91, the effect of the weight of each pulley must be estimated in each step. So also in fig. above, but in this case

the weights of the pulleys are an advantage in favour of the power.

141. From its portability and simplicity, the pulley is a most useful machine, but its mechanical advantage is diminished very considerably by friction and rigidity of the rope. It is computed, that in most cases one half of the power is lost when motion is communicated, and in some as much as two thirds.

142. The golden rule applies in this class of machines as in all others, viz. that what we gain in power is lost in velocity; this principle might have been applied to the determination of the formulæ of equilibrium.

SECTION IV.

THE INCLINED PLANE.

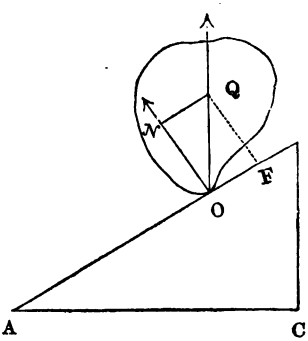
143. *The inclined plane* is the simplest of all the mechanical powers. It is merely a hard impenetrable surface inclined to the horizon.

In practice, the plane is in all cases slightly penetrable, and friction modifies the results.

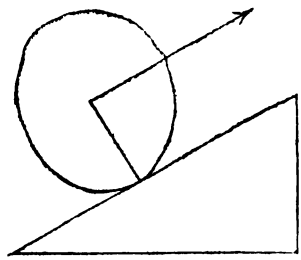
144. The advantage of the inclined plane consists in its partly supporting the weight or counteracting the effects of gravity by the reaction of its surface.

A hard plane reacts, as was remarked in art. 93. in a direction at right angles to itself. Hence

if in fig. QO represents the weight of any body placed at O or the force of gravity perpendicular to the horizon, the force OQ necessary to counteract this may arise from the combined effort of ON, OF, by the parallelogram of forces. And if ON is perpendicular to the plane, the reaction of the plane supplies a force equal to ON, and all that is required to support the weight is a force equal to OF.



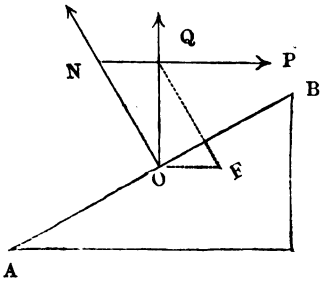
145. The power may be applied in a direction parallel to the plane, as in fig. In this case the considerations of the last art. give



P : W : : OF : OQ
: : BC : AB by sim. triangles;
: : height : length
: : sin. α. : 1.

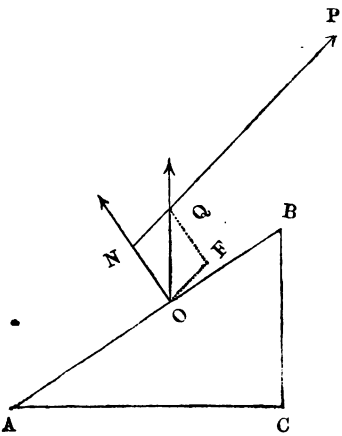
If α = the inclination of the plane to the horizon.

146. Or the power may be applied parallel to the base, as in fig.[a] In this case part of the power is against the plane, and so is lost, and by proceeding by the resolution of force we have,



$$P : W :: BC : AC$$
$$:: \text{height} : \text{base}$$
$$:: \tan \alpha : 1.$$

147. Or the power may act in any other direction, as in fig. Here proceeding as before, if $\beta =$ the angle which the direction of the power makes with the normal to the plane, we obtain



[a] In this and the next fig. the representation of the body or weight is omitted for the sake of distinctness, the directions of the forces are the main points to be attended to.

$$P : W :: \sin \alpha : \sin \beta.$$

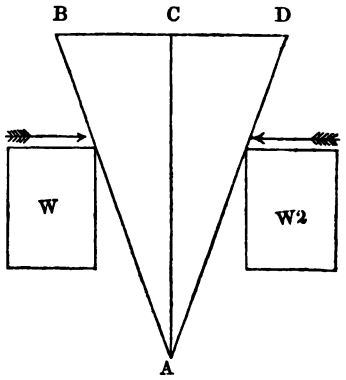
This formula includes all the cases, and may be derived at once from the principle of virtual velocities.

Inclined roads and the inclines on railways present familiar examples of inclined planes.

SECTION V.

THE WEDGE.

148. *A wedge* in practice is a moveable inclined plane, or it may be regarded as two inclined planes with a common base AC, as seen in fig.



As a mechanical power it is used thus; a force P acts on BD the back of the wedge, so as to drive it between two bodies, as W_1, W_2, and overcome whatever resistance they offer.

149. The resistance may present itself perpendicularly to the length of the wedge (CA) as in fig. above, or perpendicularly to the sides, as in

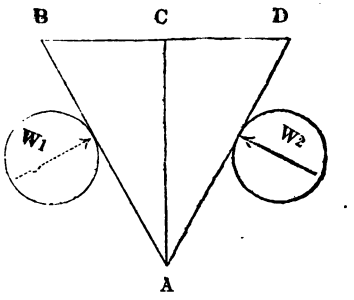


fig. p. 96. In the former case, the wedge resembles the inclined plane where the power is parallel to the base, and we have

$P : (W_1 + W_2)$ or $W :: (BC + CD)$ or $BD : CA$
:: back : length.

In the other case, (fig. above,)

$P : (W_1 + W_2)$ or $W :: (BC + CD)$ or $BD : BA$
:: back : side.

The universal rule of

P : W :: W's virtual vel. : P's virtual vel.

holds good, and may be applied here also.

150. The theory of the wedge is not however applicable in practice. The effect of friction is wholly neglected, and this bears a very great ratio to the power. Besides, the power employed is commonly *percussion*, and not pressure: and percussion and pressure are not very easily compared. We shall make some remarks on the

comparison hereafter; at present it may suffice to observe, that percussion is much more able to overcome the molecular forces which in the applications of the wedge are commonly brought into action, than almost any amount of pressure. Practically also, friction, which would be a disadvantage as opposed to pressure, becomes of use in the case of percussion, as the wedge is held in its place by friction, during the intervals of percussion.

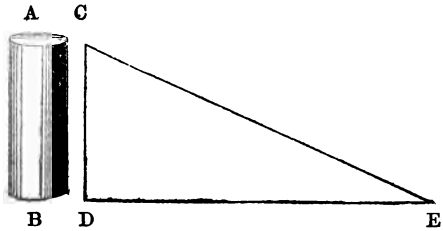
151. The stones of an arch are truncated wedges.

SECTION VI.

THE SCREW.

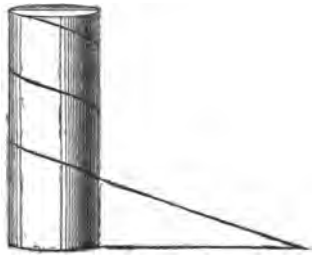
152. *The screw* is a winding wedge, or a winding inclined plane.

A *screw-line* may be thus generated. Let A B be a cylinder, (fig.) C D E a right angled triangle,



with the height C D equal to the height of the

cylinder, suppose C D to be applied to the cylinder, and then the cylinder turned so that the right angled triangle shall be rolled round it, as represented partially in fig. the hypotenuse C E will then trace a *screw*-line at the surface of the cylinder, and if a projecting thread or rib be placed wherever the screw-line is, we have the contrivance called *a screw.*



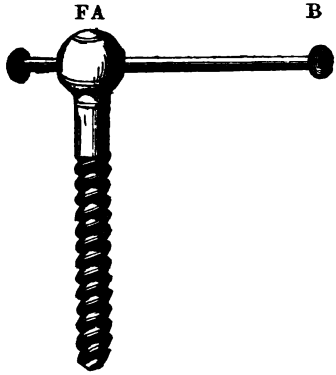
This cylinder with its projecting rib is in practice made to work in a hollow cylinder, with a spiral cavity cut out corresponding exactly to the thread on the convex cylinder. This hollow cylinder is sometimes called the *nut.*

153. This mode of description connects the screw at once with the inclined plane; and if the power be applied to the top of the screw in a direction parallel to the base, we derive as the condition of equilibrium,

If h = the distance between two threads,
and r = radius of cylinder

P : W :: height of plane : base, by art. 146.
:: dist. between the threads : circumference of cylinder
:: h : $2\pi r$

154. The most unusual way of applying the power is by means of a lever attached to the head, as in fig. In this way the screw is a compound machine, and its efficacy is determined by combining the separate efficacies; and hence, if p denote the power if applied as in last art. that is at A, and P the power at the end of the lever or at B, and FB = R,



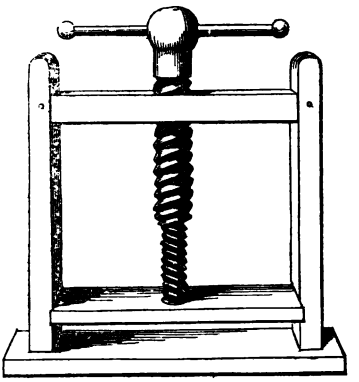
$$\therefore P : p :: FA : FB \text{ by lever}$$
$$:: r : R$$
$$\text{and } p : W :: h : 2\pi r \text{ by last art.}$$
$$\therefore P : W :: h : 2\pi R$$

or, in other words, the power is to the weight as the distance between two threads to the circumference of the circle described by the power in one revolution.

This same relation follows immediately from the universal principle of virtual velocities.

155. Power is therefore economised in the screw by making the threads very near one an-

other, or by lengthening the lever to which the force is applied. The limits are those which are caused by want of strength in the thread, and awkwardness in the length of the lever. The double screw proposed by Hunter, in which one screw works within another, one ascending while the other descends, presents a contrivance for indefinitely augmenting the efficacy of any given power.(fig.)The threads may be of any strength and magnitude, but may have a very small difference in relative distance. This machine is very similar in principle to the Chinese wheel and axle.



156. The purposes to which the screw is applied are various; it is sometimes employed

α. To compress bodies closely: as in presses, &c.

β. To raise weights, as in elevating cannon, &c.

γ. To produce uniform and delicate movements, as in micrometer screws, &c.

BOOK III.

DYNAMICS.

CHAP. I.

MOTION AND THE LAWS OF MOTION.

157. We now come to that branch of Mechanics which treats of Motion, and to which the name of Dynamics is given, because motion is an effect of force.

158. *Motion* is a change of place, as *Rest* is a permanency in the same place.

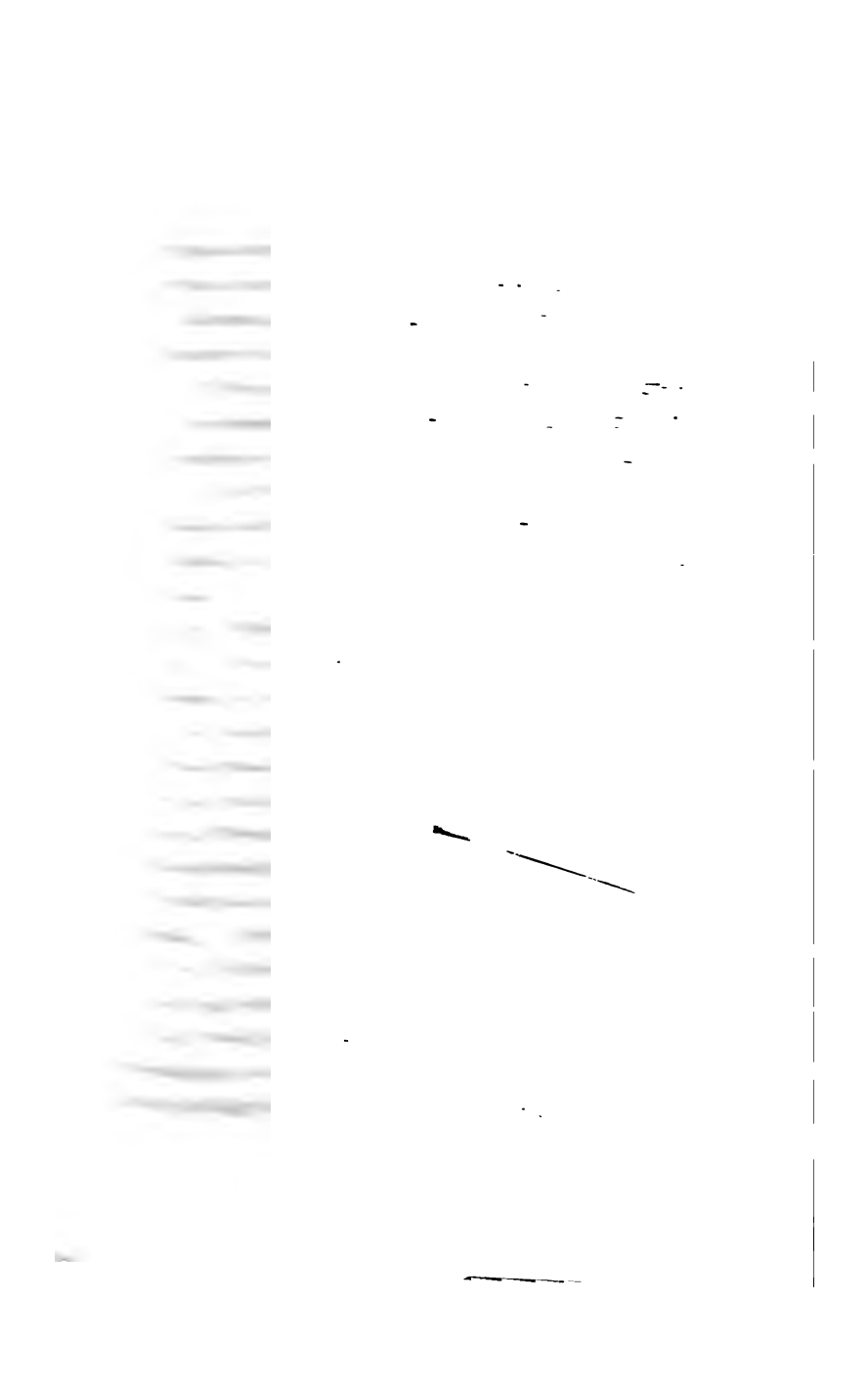
159. With regard to the motion of any body, the following points may be observed:

α. The *direction* or *path* which the body takes.

β. The *amount* of change of place, or the *space passed over.*

γ. The *time* occupied in the change.

The path in which the body moves is obviously represented by the line along which it moves, and this whether straight or curved: but in the latter case, i. e. if the path is curved, direction at any given instant would be represented by a



motion is that in whic
ng series.
niform motion, that we
y the space *actually* pas
nd in this kind of moti
through a space, S i
elocity, the three quantiti
d by the equation

$$V = \frac{S}{T}$$

from the definition of ve
niformity of motion; for
of time.
eferred to this same unit, an
pace in 1 unit :: T : 1
V :: T : 1
$S = VT$.

be *not* uniform, the velocit
measured by the space tha
the motion from that instan
during the unit of time[a].
may be either *absolute* o
may be.

ollowing will here easily deduce th
expression for the velocity in all case
&c.

The term, relative motion, is used when the motion of a body is referred to a point or body which is itself in motion; and if the bodies are moving in the same direction, the relative velocity is the *difference* of the actual velocities, but if in opposite, it is the *sum*. In both cases, the relative velocity is expressed by the formula $V_1 \mp V_2$.

165. Absolute rest is not known to exist in any part of creation. The whole universe is probably in motion, and this in consequence of an original impulse given by the will of the Creator, which remains impressed upon all matter by virtue of that great quality of inertia which was noticed in Book I.

From the inertia of matter, a motion once imparted to it must be retained until it is destroyed by some external agency; and further, if a body in motion is left to itself, it will continue to retain the imparted motion uniformly and in the same direction. This consequence of inertia is enunciated in that which is called,

166. The *First Law of Motion, viz. A body at rest will continue at rest, and a body in motion will continue in uniform rectilinear motion, unless it is acted on by some external force*[b].

[b] This law is thus stated by Sir Isaac Newton in his Principia,

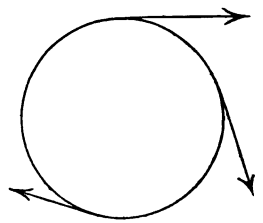
The first part of this law is self-evident. The latter, although we have considered it as a necessary consequence of the idea of matter, is not at first so readily received. Philosophers formerly held it as an axiom, that matter *loved rest;* and hence any motion imparted was soon overcome by this natural tendency to repose. The law is however capable of experimental proof, not indeed direct, but we can shew, that in proportion as the impediments to motion are removed, we approach nearer and nearer to uniform rectilinear motion. The impediments to motion on the surface of our planet are friction and resistance of the air; and these we cannot altogether remove. But in the motions of the heavenly bodies, we may have conclusive proofs of the law. Every movement of every part of that vast and wonderful system can be explained on the supposition of the truth of the law, and cannot be explained without. Each body rolls on in its appointed path, fulfilling its course, because by its inertia it retains the impulse which launched it into space.

"Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare."

But we may trace the effects of this law in our daily experience. A large body cannot be stopped suddenly, e. g. a railway train. Again, in a sudden collision, the passengers are thrown forward.

Fly-wheels in machinery derive their use from this law.

167. An important effect of this great law is seen in all those phænomena which are said to be the effects of *Centrifugal Force*[c]. Whenever a body is compelled to move in a curve, it has from its inertia a tendency to fly off from the curve at a tangent, as represented in fig. and it would do so unless it were retained in the curve by some force, whether it be a string, or some force of attraction. This tendency is capable of exact estimation, and in a subsequent chapter its measure will be given; but it is an immediate consequence of the great property of inertia, as stated in the first law of motion.



[c] The term *force* thus employed is obviously inaccurate, and so also the expression *vis inertiæ*.

Many interesting and important phænomena arise from this tendency, or, as it is called, centrifugal force.

168. *Motion* has been stated to be the effect of force, and *force* we defined (art. 46.) to be that which produces or tends to produce motion, or a change of motion. This mode of statement restricts the idea of force to its effects, and this is the best method of proceeding, as we are commonly ignorant of any cause, except so far as it is made known by its effects.

But there is a difference in the way in which forces act, and a corresponding difference in their effects: and it is usual to divide forces with regard to the duration of their action into *momentary* or *instantaneous* and *continued* or *permanent.*

169. *Momentary* forces are those which act only to set the body in motion, and no longer exert any influence, e. g. an impulse or blow.

170. *Continued* forces exert their influence during the motion without interruption or cessation, as is seen when a stone is drawn to the ground by the force of gravity.

171. In strictness, momentary forces require a time for the production of their effect, although

this time is very small; and impact and pressure may be brought more nearly into comparison, by regarding impact as a force of great intensity acting during a very small time, whereas pressure is a force of small intensity acting during a long time. There is nevertheless a difficulty in making the comparison. Impact produces effects wholly different from those which are produced by pressure, overcoming by the suddenness of its energy the state of rest of the molecular forces of a mass, or the friction between different bodies.

Momentary forces having acted on a body, leave it to itself, and hence the body by its inertia moves off with the velocity imparted, and this velocity is a measure of the force.

172. *Continued* forces are divided into *uniform and variable.* An *uniform continued* force is that which produces either an *uniformly accelerated* or an *uniformly retarded* motion; and the same force may be either accelerating or retarding, according as it acts in the direction of motion, or in an opposite direction. The force which produces an uniformly accelerated motion is sometimes called an *uniformly accelerating* force, or more concisely, an *uniform force.*

It is measured by the velocity which is produced in a given time, which in scientific investigations is generally one second: and since in consequence of its uniformity it produces velocities always proportional to the time of its action, we have a simple relation between the force f, the velocity produced v, and the time of action t, assuming t to be expressed in seconds.

For, vel. in $1'' : v :: 1'' : t$

$$\therefore \text{vel. in } 1'' \text{ or } f = \frac{v}{t}$$

$$\text{or } v = ft.$$

173. In *variable* forces we must ascertain the velocity that would be produced if the force could be continued uniform, or obtain the increment of velocity in the indefinitely small increment of time[d].

174. This method of estimating forces by their effects is equivalent to the statement contained in that which is called

The second Law of Motion, which is usually enunciated thus: *Every motion or change of motion produced in a given body in a given time*

[d] This is expressed mathematically $f = \frac{dv}{dt}$.

is proportional to the force which produces it, and in the direction of the force[e].

This law of motion is involved in the definition of force, and therefore it ought not to be considered as a law, but rather an enunciation of results[f].

175. It may be remarked, however, that this law of motion contains a distinct statement, that the law of the composition of velocities is identical with that of the composition of pressures, and that we may apply our conclusions as to the composition of statical forces in Book II. Chap. I. to the composition of motions. This is indeed no more than necessarily follows, but it must be observed for the sake of accuracy of reasoning.

176. The above estimates of the effect of a force do not take into account the *mass* or *quantity of matter* which is set in motion, or, in other words, we estimate force by the velocity generated in the unit of mass. When the mass is

[e] Newton's enunciation of the law is this: "Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur."

[f] This question is most ably and thoroughly discussed in a Paper by Professor Powell, on the *Nature and Evidence of the Primary Laws of Motion*, published by the Ashmolean Society.

taken into account, the force is called a *moving force*, and a moving force is estimated by the *product of the mass and the velocity generated;* or, moving force = mass × vel. generated.
= mass × acc. force.

The product of the mass and velocity is called the *momentum* of the moving body, and sometimes the *quantity of motion;* but this term is liable to be misunderstood.

177. This definition of momentum enables us to give a simple explanation of

The third law of Motion, viz. *action and reaction, are equal and opposite*[g].

This law may be thus stated; *in the mutual action of two bodies, the momentum lost by one is equal and opposite to that gained by the other.* In fact, the word *action* in the law means moving force.

This law, like the second, is sometimes regarded as a necessary consequence of the definitions: it is, however, capable of experimental illustration, and it expresses a law of great importance in determining the effects of collision, or the mutual action of bodies on each other.

[g] Newton's enunciation is, "Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi."

CHAP. II.

ON RECTILINEAR MOTION.

178. In the last chapter we explained the terms *uniform motion*, *uniformly accelerating force*, &c.

The simplest case of rectilinear motion is that which would result from a momentary or impulsive force, if no other force or no impediment interfered with the effect. The first law of motion expresses the consequence. But we have rectilinear motion arising from the action of continued forces, and the relations of force, time, space, and velocity, in the case of an uniformly accelerating force, are of importance.

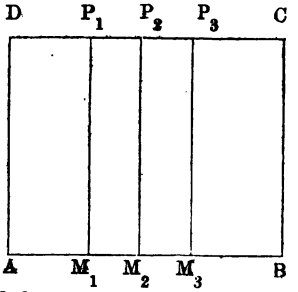
179. PROPOSITION. *The space described from rest by a body under the influence of an uniformly accelerating force is, at the end of any time, half that which would be described if the body had moved for the same time with the last acquired velocity continued uniform during the whole time.*

It has been shewn, art. 162, that in uniform

motion the space (s) the velocity (v) and the time (t) are connected by the equation

$$s = t\,v.$$

Hence if one side AD of the rectangle ABCD, (fig.) represent the velocity, and the adjacent side AB represent the time, the rectangle ABCD will represent the space described: i. e. it will have the same number of units in it as the space would have according to the proper units of time and velocity; for

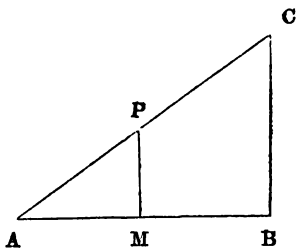


$$\text{the rectangle} = AB \times AD$$
$$= tv.$$

Also it is evident, that if AM_1 represent any intermediate time, the perpendicular M_1P_1 will be the velocity at that time, and for every AM, we must have MP = AD or the velocity constant.

But if the body move from rest by the action of an uniformly accelerating force, the velocity at the commencement is zero, and it goes on increasing uniformly during the whole time of motion, and by the definition, art. 172, it is pro-

portional to the time. If therefore AB the base of the right angled triangle ABC (fig.) represent the time, and the perpendicular BC represent the last acquired velocity, the triangle itself will represent the space, since at any intermediate time as that which would be represented by AM, the perpendicular MP would represent the velocity at that time, for



$$AB : BC :: AM : MP.$$

But the triangle $ABC = \frac{1}{2} AB \times BC$

$$\therefore \quad s = \tfrac{1}{2} tv$$

where $v =$ *the last acquired velocity*[s].

180. By combining this formula with that deduced in art. 172, viz.

$$v = ft$$

we obtain

$$s = \tfrac{1}{2} t \times ft$$
$$= \tfrac{1}{2} ft^2.$$

Therefore with the same force, *the spaces, reckoning from rest, are as the squares of the times.*

[s] This proof is not so complete as that given by Wood in his Mechanics, in which the method of Newton is adopted : but it may be regarded as sufficient in a work of this kind.

181. Also as $t = \frac{v}{f}$

we have $s = \frac{1}{2} v \frac{v}{f}$

$$\therefore s = \frac{2v^2}{f} \quad \text{or } v^2 = 2fs.$$

Hence with the same force *the spaces from rest are as the squares of the last acquired velocities.*

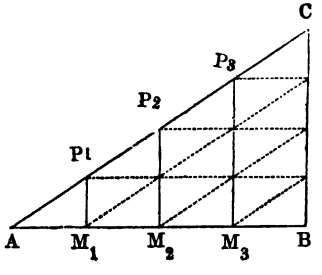
These two conclusions might have been at once deduced from the representation of the space by the triangle ABC, as in art. 179; for by Euclid, the areas of similar triangles are as the squares of the sides,

Hence area of ABC $\propto$ AB^2 or $\propto$ BC^2.

$$\therefore \quad s \propto t^2 \text{ or } \propto v^2.$$

182. PROPOSITION. *The spaces described from rest in equal successive portions of time by a body acted on by an uniformly accelerating force, are as the series of odd numbers,* 1, 3, 5, 7, &c.

This is readily deduced from the formulæ of art. 180; but it is also evident from a simple inspection of the fig. in which if $AM_1 = M^1M_2$



$= M_2M_3 = M_3B$, and each represents equal times, the areas AM_1P_1, $P_1M_1M_2P_2$, $P_2M_2M_3P_3$, P_3M_3BC, will on the principle of art. 179. represent the spaces; and it is plain that

$$P_1M_1M_2P_2 = 3AM_1P_1$$
$$P_2M_2M_3P_3 = 5AM_1P_1$$
&c. &c.

183. The best exemplification of these formulæ is obtained from the motion of bodies falling from the action of gravity.

The property of gravity, as possessed by all bodies on the surface of the earth, was noticed in Book I. This property is indeed only a peculiar effect of an universal principle called Gravitation, by virtue of which one particle of matter has an attraction for and is attracted by every other particle. It is this principle which sustains the earth and the planets in their orbits, and governs their motions. The several bodies of our system are attracted by the sun and by each other, and every circumstance of their motions, every irregularity and perturbation, is deducible from this one principle of gravitation, combined with an original impulse from the great Creator.

The experiments of Dr. Maskelyne, near Skehallion, and the very delicate observations of Cavendish, have clearly shewn, that this principle

acts in and upon every particle of matter. It is this which causes the apple to fall when liberated from the tree, and determines the revolutions of the moon.

184. The law according to which one body attracts another is ascertained to be the mass of the attracting body divided by the square of the distance from its centre; or expressed thus,

$$\text{Force} \propto \frac{\text{mass}}{(\text{dist.})^2}$$

Hence if the distance be the same in two cases, or when two attracting bodies are compared, the force will simply vary as the mass. Thus if one of Jupiter's moons be supposed to be at the same distance from his centre as our moon is from the centre of the earth,

The force of Jupiter on his moon : The force of the Earth on our moon :: mass of Jupiter : mass of the Earth.

Also if the same attracting body act on bodies at different distances, the mass is the same and the force varies inversely as the square of the distance. Thus the two nearest of Jupiter's satellites are at distances from him which are approximately represented by 6 and 9; hence taking these as the distances,

$$\text{The force on 1st satellite : The force on second} :: \frac{1}{36} : \frac{1}{81}$$
$$:: 81 : 36$$

185. When a stone is liberated at a distance from the surface of the earth, in mathematical strictness it attracts the earth, and the earth attracts it, but on account of the great mass of the earth compared with that of the largest stone that can be placed at a distance from it, the space through which the earth is drawn is absolutely inappreciable. The stone therefore descends, and it *strictly* descends in a line which if produced would pass through the centre of the earth. The directions in which the force of gravity thus acts at different places are truly lines converging to the centre, and a plumb line is such a line. But if two plummets were suspended at a distance of 1200 feet from one another, the deviation of the lines from parallelism would not exceed the 240th part of a degree. When therefore we are speaking of the descent of a body near the surface, or comparing the motions of bodies near one another, we may assume that the force of gravity acts in lines parallel to one another, and perpendicular to the horizontal plane or surface of still water.

186. Again, the force of attraction will have a different intensity on bodies at different distances, since it will vary inversely as the square of the distance from the earth's centre. But if we com-

pare the force on bodies at a difference of altitude of one mile or half a mile, the difference of the force is very small. For instance, if we take the radius of the earth as 4000 miles,

The force at surface : force at ½ mile above : : 64,016,000 : 64,000,000
: : 401 : 400

187. There is also a variation in the force of gravity, arising from centrifugal force. A body at the equator has a greater tendency to fly off from the earth than in any other position, for at the equator the radius of the circle in which it revolves is greater. The law of this centrifugal action will be given hereafter.

188. At a given place, however, the force of gravity acting on bodies near the surface of the earth may be taken as constant, and since it acts equally and independently on all the particles which compose a body, it has a tendency to make all these particles move with equal velocities, and in parallel lines perpendicular to the surface. Hence it follows, that all bodies are equally accelerated by the force of gravity. This important conclusion, which was first asserted by Galileo, seems at first sight to be contradicted by the slower descent of light bodies; but this is owing to the resistance of the air. Experiment shews, that

in vacuo light and heavy bodies will fall together with a common velocity. But we may arrive at the conclusion by the following consideration.

It is easily conceivable, that if two equal balls of the same weight were placed side by side at the same height, and both were liberated at the same instant, they would descend together, and would strike the ground at the same instant with equal velocities, and side by side. In the same way, if any number of equal balls, say ten or one hundred, were placed side by side, and set free at the same instant, they would fall together, and no difference would appear if two or more or even all were joined together. They would descend in the same way, and reach the ground at the same time and with the same velocity were they all or any of them joined or connected in groups. Nor would there be any different result if any number were moulded into one ball. The larger ball would fall as the smaller, since form would not affect the operation of gravity.

189. Gravity at the same place is an uniformly accelerating force, and the velocity communicated in any time, and the spaces passed through, are connected by the equations in arts. 179, 180, 181,

&c. In order to apply these formulæ, we require the determination of the value of f, or the force of gravity. This we can determine if we know how far a body will fall in the unit of time, or in 1″.

It appears by experiment[h], that in the latitude of London, a body will fall in vacuo through 16.0904 feet in 1″, and since

$$s = \tfrac{1}{2} f t^2$$

we have space in $1'' = \frac{1}{2} f$, or $f = 2 \times$ space in 1″. Hence for the latitude of London,

The force of gravity or $f = 32.1908$ feet.

This is generally represented by g, and our formulæ then become

$$v = gt; \qquad s = \tfrac{1}{2} g t^2; \qquad v^2 = 2gs.$$

For convenience of illustration, we may omit decimals, and assume the value of g as 32 feet; and then we may exhibit the velocities and spaces in different times in the following Table.

[h] This experiment will be mentioned when the vibration of a pendulum is discussed. See Chap. V.

Time.	Velocity acquired.	Space passed through.
1″	32 feet	16 feet
2″	64	64
3″	96	144
4″	128	256
5″	160	400
6″	192	576
&c.	&c.	&c.

If we subtract the numbers in the last column, each from the one below it, we have the spaces passed over in the several successive seconds, and then we have

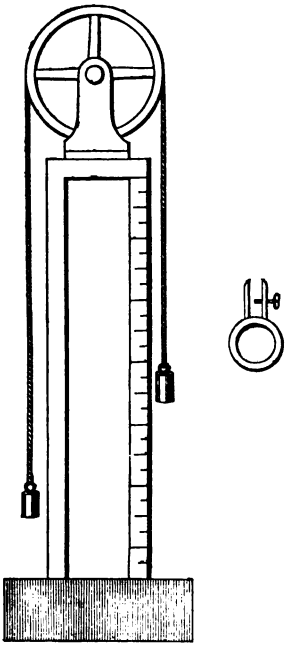
space in 1st second $= 16 = 1 \times 16$
space 2nd $= 64 - 16 = 48 = 3 \times 16$
space 3rd $= 144 - 64 = 80 = 5 \times 16$
space 4th $= 256 - 144 = 112 = 7 \times 16$
space 5th $= 400 - 256 = 144 = 9 \times 16$
&c.

where the spaces are as the odd numbers 1, 3, 5, 7, &c.

190. An experimental exhibition of these results of calculation would be impossible not only from the great height required, but from the resistance of the air. The machine of Atwood enables us, however, to diminish the force

of gravity in any proportion, and to make the falling body descend at so slow a rate, that the resistance of the air is inappreciable. By this machine all the formulæ may be verified.

The principle of the machine is simply this: Two equal cylinders of the same weight W are connected by a silk cord passing over a wheel, as in fig. The cord is supposed perfectly flexible, and the wheel so constructed as to get rid of friction. The two equal weights are therefore in equilibrium, and will rest if placed at rest, but will move freely and uniformly according to the first law of motion, if any motion is given.



But if a small additional weight is added to either side, the weight on that side will descend

drawing up the other, and it will descend with all the characters of an uniformly accelerating motion, but in consequence of the weight on the other side its descent will be slow.

In fact, if we denote the mass of each cylinder by M, and the mass of the small weight by m, and call f the force which causes the descent, we have by art. 176.

The moving force = mass moved × accelerating force

$$= (M + M + m) f,$$
$$= (2M + m) f.$$

But the moving force of the descending body, if not retarded by the other, would $= (M + m)\, g$

and the retarding moving force $= Mg$.

Hence the moving force would equal the difference $= mg$;

$\therefore$ Equating these two values, we obtain,

$$(2M + m) f = mg$$
$$\therefore f = \frac{m}{2M + m} g.$$

By making m small compared M, f may be any fractional part of gravity; for example, if $m = \frac{2}{99} M$ it will be found by substitution that

$$f = \frac{1}{100} g.$$

Hence the effects will be the same as if the force of gravity were reduced to the $\frac{1}{100\text{th}}$ part; and while a body would fall in vacuo through 100 feet, the weight to which w has been added will descend through 1 foot, and so on. We have then by this machine a miniature representation of the phænomena of falling bodies.

In order to simplify the explanation of this machine, the effect of the inertia of the wheel over which the cord passes has been omitted, this will in some degree affect the motion, and when accuracy is required, must be taken into account. It will however be but a small quantity, and it will have to be added to the mass set in motion.

When the machine is used to verify the formulæ, it is provided with a graduated vertical scale; the small weight which is added to cause one cylinder to descend is in the form of a small metal bar which is laid on the cylinder; this as it descends is caught by a metal ring, which is so contrived as to slide up and down the scale, and may be fixed in any required position. The sound of the small bar striking the ring marks the time of descent, and enables us to compare it with the beats of the pendulum. Further also,

the descending cylinder when relieved of the small weight will move on with the velocity acquired, and we may measure this by placing a stage at any point of the vertical scale, so that the cylinder may strike it.

CHAP. III.

ON PROJECTILES NEAR THE EARTH'S SURFACE.

191. THERE are two forces in operation on every projected body, by whatever means it may have been projected; viz. the projectile force, which is momentary or instantaneous; and the force of gravitation, whose action is continued. The motion thus obtained is a compound one, and the results will be different according as these forces are

α. In the same direction.

β. In exactly opposite directions.

γ. Or one acts in a direction inclined to the other, whether the angle of inclination be a right or an oblique angle.

192. If the projectile force acts in the direction of gravity, as when a stone is thrown vertically downwards instead of being simply liberated, the path of the body is obviously in the vertical, but its accelerated motion is increased at each moment by the velocity communicated at the instant of projection. Thus if v_1 be the velocity of pro-

jection, its velocity at any time (t) afterwards is given by

$$v = v_1 - g\,t.$$

So also the space described is the sum of the spaces *due* to the projectile velocity and the force of gravity, or

$$s = v_1\,t + \tfrac{1}{2}\,g\,t^2.$$

193. If the body be projected vertically *upwards*, gravity becomes an *uniformly retarding force*, and continually diminishes the projectile velocity until it has entirely destroyed it. The body therefore will only continue to ascend for a certain time, after which, having lost its upward velocity, it will begin to descend just as a body let free, and it will return to the earth in just the same time that was occupied in the ascent. Also, at each point of the descent the velocity will be exactly equal to that which it had at that point while ascending, but in the opposite direction.

The velocity at any point of its ascent will in this case be given by the formula

$$v = v_1 - g\,t,$$

and the time of ascent, or time to the point of rest, will be found by putting $v = 0$

$$\text{or } v_1 - g\,t = 0 \quad \therefore v_1 = g\,t \text{ or } t = \frac{v_1}{g}.$$

The position of the body will be determined by

$$s = v_1 t - \tfrac{1}{2} g t^2.$$

Hence we may find the height to which the body will rise, and by substituting in this,

1st, The value of v_1 we have $s = \tfrac{1}{2} g t^2$.

2dly, The value of t we have $s = \tfrac{1}{2} \frac{v_1^2}{g}$

or the height to which it will rise is the same as that through which it would fall freely in the same time; also it is equal to that through which it would fall freely to acquire the velocity of projection. Hence also, the body at the end of its fall has a velocity exactly equal to that with which it was projected, though in the opposite direction.

194. The formulæ of the last arts. may be exhibited in one view by the use of the double sign $\pm$, thus,

$$v = v_1 \pm g t \qquad s = v_1 t \pm \tfrac{1}{2} g t^2,$$

the upper or lower sign being taken according as the body is projected vertically downwards or upwards.

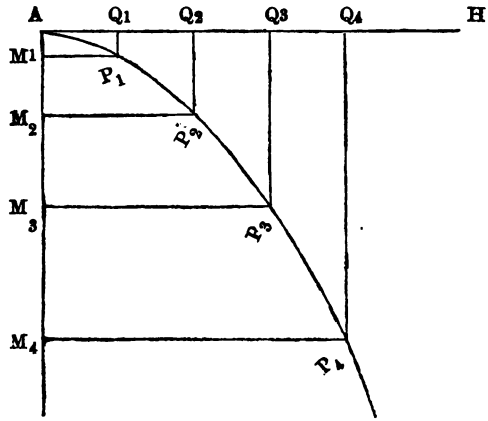
195. The general formulæ for rectilinear motion, when the force is uniformly accelerating, may be applied similarly when a body is projected in

the same or opposite direction, by merely putting f for g; thus,

$$v = v_1 \pm f t \qquad s = v_1 t \pm \tfrac{1}{2} f t^2.$$

196. If a body be projected so that the line of projection is oblique to the force of gravity, the body will move in the direction of neither force, but will take a path between the two, according to the relative intensity of the forces. (art. 56.) But as the attraction of the earth is continually acting, the path will not be rectilinear, but a curved line. This will be seen, if we take the simplest case first.

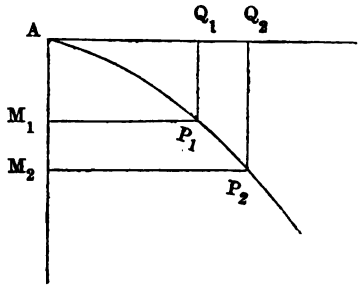
Suppose the body projected horizontally in the direction A H, (fig.) then if the projectile force



alone acted, the body would describe along this line equal spaces in equal times. Let AQ_1 represent the space along this line described in 1″ of time, then if equal spaces be taken so that $AQ_1 = Q_1Q_2 = Q_2Q_3 =$ &c. the body would if undisturbed be at Q_2 at the end of the 2d second, and so on. Now draw AK in the vertical direction, or direction of gravity, if we measure on this $AM_1 = 16$ feet, and take $M_1M_2 = 3\,AM_1$, $M_2M_3 = 5\,AM_1$, $M_3M_4 = 7\,AM_1$, and so on: the body if let fall would be at M_1, at the end of the 1st second, at M_2 at the end of the 2d, at M_3 at the end of the third, &c. Hence if we complete the parallelograms $AM_1P_1Q_1$, $AM_2P_2Q_2$, &c. as in fig. when both forces act at once, the body will be found at the end of the first second at P_1, in 2″ at P_2, in 3″ at P_3, and so on; but as the force of gravity is a continually accelerating force, the velocity due to this will be continually increasing, and the body will not have moved through the diagonals of the parallelograms, but will have traced a continuous curve instead of a broken line.

197. This curve will be the parabola with its vertex at the highest point or at A, and it may be thus shewn that the curve is of this form.

It is a property of this curve, that any abscissa as AM_1 (fig.) is to any other abscissa AM_2 as the square of the corresponding ordinate M_1P_1 is to the square of M_2P_2 : and this is the relation between the abscissa and ordinates of this curve, for



if t_1 = the time from A to P_1

t_2 = A to P_2

we have $AM_1 = \frac{1}{2} g t_1$, $AM_2 = \frac{1}{2} g t_2^2$

or $AM_1 : AM_2 : t_1^2 : t_2^2$

but if v_1 = the velocity of projection,

$$M_1P_1 = AQ_1 = v_1 t_1$$

$$M_2P_2 = AQ_2 = v_1 t_2$$

$$\therefore M_1P_1 : M_2P_2 :: t_1 : t_2$$

$$\therefore AM_1 : AM_2 :: M_1P_1^2 : M_2P_2^2.$$

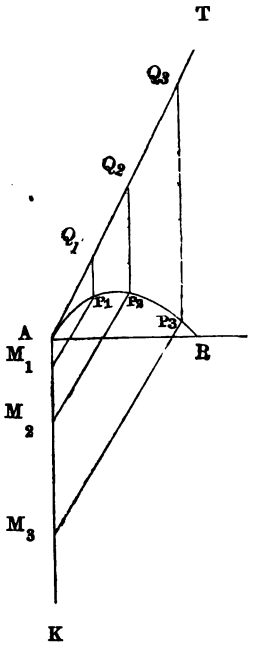
Hence *the path of the projectile is a parabola with its vertex at the highest point.*

From this investigation we may derive the two following conclusions.

198. *A body projected horizontally, has a horizontal motion equal to that which it would have if acted on by the force of projection alone.*

199. *A body thus projected descends towards the earth in the same time as it would have done if had fallen freely from the same height.*

200. PROPOSITION. *If a body be projected obliquely upwards, its path will be a complete parabola, whose vertex will be at the highest point to which the body ascends.*



The path will be obtained in a manner very similar to that in the last article. Let AT (fig.) represent the direction of projection, take on this equal lengths $AQ_1 = Q_1Q_2 = Q_2Q_3$ and to represent the spaces *due* to the velocity of projection in equal times, draw AK in the vertical direction, and let AM_1 = the space due to gravity in the time (t) in which AQ would be described; measure $M_1M_2 = 3\,AM_1$, $M_2M_3 = 5\,AM_1$, &c. as before, and complete the paral-

lelograms, the path of the body will be the curve $AP_1P_2P_3 \ldots$. and it may be shewn, as in the last, that $AM_1 \propto M_1P_1^2$, &c.

201. In these investigations it is assumed, that the resistance of the air does not affect the motion, or that the projectile moves *in vacuo*. In practice the resistance of the air causes the body to deviate considerably from the true parabola, and more and more, as the velocity of projection increases. The determination of the path of a projectile in the air, is a problem which has never yet been solved. The resistance of the air within certain limits of the velocity varies as the square of the velocity, but assuming this law, the path can only be determined approximately.

CHAP. IV.

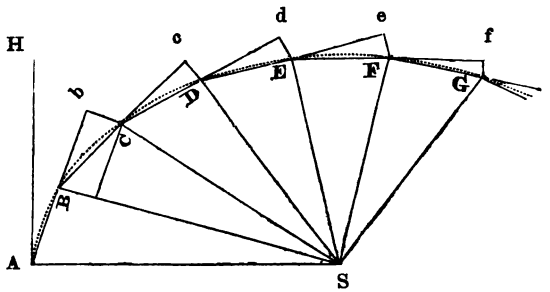
ON MOTION ROUND A CENTRE OF FORCE.

202. THE principles which we have employed in the last chapter, will enable us to present some of the great laws which govern the motion of the celestial bodies. We assume that the planets are retained in their orbits by the force of gravitation. The sun from its superior mass draws them towards itself. They in their turn draw or attract the sun, but they compared with the sun are so small, that this part of the action may be neglected. The effect of the sun is therefore that of a *centripetal force*, constantly attracting the planet towards it. An impulse is communicated (we may suppose) to the planet, which gives it a velocity in a direction oblique to the line drawn from its position to the sun. The result of these two forces will be a curvilinear motion, and the path will be such as to fulfil the following

203. PROPOSITION. *If a line be drawn from the sun to the initial position of the planet, and this line be supposed to accompany the planet in its revolution, it will sweep out sec-*

torial areas which are all in the same plane, and such that equal areas are swept out in equal times.

The force acts continually, but we may suppose, (at first,) that it acts not continuously, but by sudden pulls at the end of equal successive portions of time; and let this time (t) be finite. Suppose then that the planet is projected in the direction AH, (fig.) and that S represents the



position of the sun. The sun acting on the planet causes it to move *not* in the direction of projection, but in some line within the angle SAH as AB, and suppose AB to be the space thus described by the planet in the time t. The sun by hypothesis having acted at A does not act again until the end of the time; if then there were no interference, the planet would in the

next interval of time describe $Bb = AB$ in the same straight line with AB, but at B the sun gives another pull in the direction BS, and in consequence of these two forces in Bb and BS, the planet will move in some direction within the angle bBS. Suppose BC this direction, then the point C at which the body will arrive in the next interval t, will be found accoording to the parallelogram of forces by drawing bC parallel to BS. By the same reasoning, if the sun did not act when the planet arrives at C, it would describe $Cc = BC$ and in the same line, but the sun pulls it towards itself, the planet therefore moves in some direction within the angle cCS as CD, and D is found by drawing cD parallel to CS. The reasoning may be continued, and on the supposition we have made the planet would describe the polygonal path ABCDEFG.

Now every part of this polygon can be shewn, by Euclid XI. to be in the same plane. Also by Euclid I.

$$\text{triangle } SAB = \text{triangle } SBb^{a};$$

for they are on equal bases, and between the same parallels, and

$$\text{triangle } SBb = \text{triangle } SBC,$$

[a] The line Sb is not drawn in the diagram.

for the same reason as bC is parallel to SB;

$$\therefore \text{triangle } SAB = \text{triangle } SBC$$

$$\text{and so} \quad = \text{triangle } SCD$$

$$= \text{triangle } SDE$$

&c.

Hence taking any number together in groups we have

area SABC : area SDEFG :: time from A to C : time from D to G

or areas proportional to the times.

But the same conclusions will be true if we suppose the intervals of the action of the sun to be diminished, and so however much they may be diminished; and if we go on diminishing the intervals of the pulls, we approach nearer and nearer to continuous and uninterrupted action: and the *limit* of our false hypothesis is the true case of the action of the sun. But the limit of the polygonal figure is a curve, represented in the figure by the dotted line, possessing the same properties; and hence the proposition is proved.

204. This very important proposition was first demonstrated by Newton[a], and as above, only more concisely. Kepler arrived at the conclusion from observation, and it is usually enunciated as the *first* of Kepler's laws.

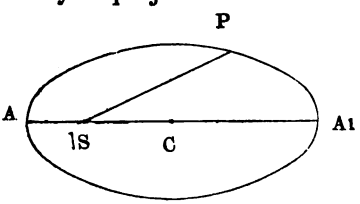
[a] Principia, Prop. I.

205. Kepler's *second* law is expressed in the following

PROPOSITION. *The orbits of the planets are ellipses, with the sun in one of the foci.*

This was deduced by Kepler from observation. It is proved by Newton[b], that if the orbit is an ellipse with the sun in one focus, the planet must be attracted by a force which obeys the law of gravitation, viz. that of the inverse square of the distance: and *conversely*[c], that if such is the law of centripetal force, the path described by a body projected from a given point with a given velocity and in a given direction, will be a conic section, i. e. an ellipse, hyperbola, or parabola, according to the velocity of projection.

206. The law may be exhibited more clearly by reference to fig. where APA_1 represents an elliptic orbit, S the Sun in one of the foci, P the position of a planet. Then



$$\text{the force on P} \propto \frac{1}{SP^2}.$$

[b] Principia, Prop. XI. [c] Principia, Prop. XVII.

The major axis ACA_1 of the ellipse is sometimes called the line of *apsides*.

The vertex A nearest to S is the perihelion or nearer apse.

The other A_1 farthest from S is the aphelion or farther apse.

207. Since by art. 203, equal areas are swept out by SP in equal times, the velocity of the planet is the greatest at perihelion, and least at aphelion.

208. The *third* law of Kepler is expressed as follows:

PROPOSITION. *The squares of the periodic times, or the times of a whole revolution of the planets, are to one another as the cubes of the major axes of their orbits.*

This remarkable relation was deduced by Kepler by a process purely tentative. A simple demonstration is given by Newton[d].

[d] Principia, Prop. XV.

CHAP. V.

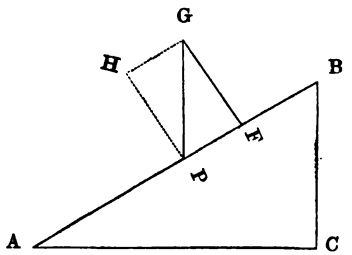
ON CONSTRAINED MOTION.

209. HITHERTO we have considered the motion of a material point, or a body conceived as a point, either as it is undisturbed by the action of any forces, or simply subjected to the action of an accelerating, retarding, or deflective force. We now proceed to examine the effects of an additional modification, which is introduced when the motion is limited to a given line or surface of any kind; the body being supposed to *slide* on a surface, or confined to an imaginary curve or surface by a thread or threads. Suspension by a thread is the most convenient way of making experiments on constrained motion, because it gets rid of the effects of friction; and the rigidity of the thread or rope is almost inconsiderable in small motions: but it is not always easy to confine the body by threads to a given surface, and impossible thus to limit it to a perfect plane.

210. PROPOSITION. *When a body is placed on an inclined plane, the force urging it to descend in*

the direction of the plane is to the force of gravity as the height of the plane is to the length.

We have already seen, (arts. 144, 145.) that the reaction of the plane partly neutralizes the effects of gravity, and in art. 146, it has been shewn by the resolution of force, that the power necessary to support a body on a plane, when the power acts to the greatest advantage is in the ratio of the height of the plane to the length. Hence the force acting down the plane is in this ratio. This is also evident from the inspection of the fig. where if GP represents the force of gravity, GH = FP represents the portion of it which is effective down the plane, the other component GF is neutralized by the reaction of the plane. And by similar triangles,



$$FP : PG :: BC : AB$$
$$:: \text{height} : \text{length}.$$

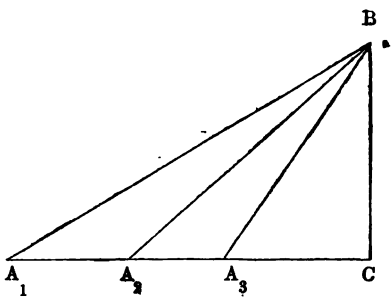
Hence if f = the force down the plane, and g = the force of gravity as before;

Also, if h = height of the plane
l = length
and α = the angle of inclination
$$f : g :: h : l$$
$$\therefore f = g\frac{h}{l} = g \sin \alpha;$$ by Trigonometry.

The force therefore is an uniformly accelerating force, and the relations of time, space, and velocity are determined by substituting this value of f in the formulæ of art. 179, &c.

211. It follows from these formulæ, that when bodies descend on any inclined planes of equal heights, but of different inclinations, the times of descent are as the lengths of the planes, and the final velocities are equal, though in different directions.

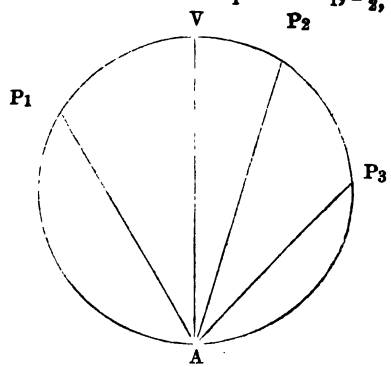
For instance, if BA_1, BA_2, BA_3, (fig.) represent three planes with the same height BC, the times of a heavy body down them are as the lengths, but the velocities at the several points A_1, A_2, A_3, are all equal.



212. There is an elegant proposition of a similar nature, which may be thus enunciated.

PROPOSITION. *If the plane of a circle be vertical, and any number of chords be drawn from points in the circumference to the lower extremity of the vertical diameter, the times of falling through all those chords are equal.*

Thus if V A be the vertical diameter of the circle P_1VP_2A, and P_1A, P_2A, P_3A represent any chords drawn to A the lower extremity, bodies let fall at the same instant from the points P_1, P_2, P_3, down the chords will arrive at the point A at the same instant. This property of the synchronous descent down the chords of a circle is easily deduced from the formulæ by help of a common property of the circle.



213. The force down the inclined plane is, art.

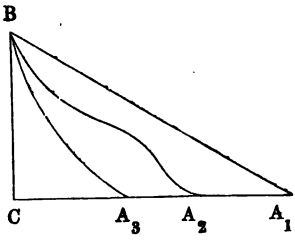
210, in a ratio to that of gravity, dependent on the inclination of the plane: hence by varying that inclination we may obtain a force which shall be any required portion of the force of gravity.

Thus if the inclination of the plane be 30°, or the ratio of height to length : : 1 : 2, we have a force $= \frac{1}{2} g$.

By a less inclination, where $h : l : : 1 : 12$, we obtain a force $= \frac{1}{12} g$, and by such a force the effects of gravity might be exhibited on a small scale as well as by Atwood's Machine, art. 190, but the results are interfered with by *friction*. When a body slides along a surface, there is a loss of force which may be very considerable from the rubbing of the surfaces, or from the production of a rotatory motion, or from both these causes combined. The consideration of these effects is reserved to a subsequent chapter. See Chap. IX.

214. The conclusion, art. 211, with regard to the velocity acquired down a plane, applies also to every surface or curve, whether the body move down the curve, or be retained in it by a thread; in all cases, the velocity is the same as that of a body falling freely down the same perpendicular height.

Thus in fig. if a body descends from B, its velocity in reaching the horizontal line CA_1 will be the same, whether it falls vertically down BC, or moves along any of the lines BA_1, BA_2, BA_3.

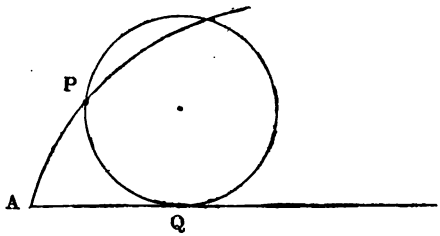


Hence if a body descend down a curve and move up a curve on the other side with the velocity acquired, it will ascend to a height equal to that from which it has descended.

215. The investigation of the time of descent of heavy bodies down curves, is a problem not always capable of solution; but there is one curve, viz. *the cycloid*, for which the solution of the problem is easy, and which possesses some remarkable and important properties.

216. The *cycloid* is the curve traced by a given point in the circumference of a circle, while that circle rolls upon a straight line.

Thus if in fig. PQ is a circle rolling on the straight line AQ, and P a point in the circumference, the curve AP traced by P is part of a cycloid; A is the point where P would be, if the circle rolled back.



Hence as the arc PQ has unrolled, as it were, from the straight line, the arc PQ = the line AQ.

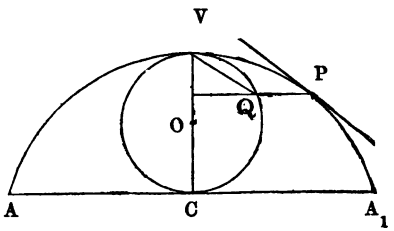


Fig. represents a whole cycloid or curve traced by one revolution of the circle, which is called the *generating circle.*

The line AA_1 is called the *base*, and for the reason above, the base AA_1 = the circumference of the generating circle.

The line VC which bisects AA_1 at right angles is called the *axis*, and evidently = the diameter of the circle.

217. The curve is one of the few curves which are *rectifiable*, or whose precise length can be

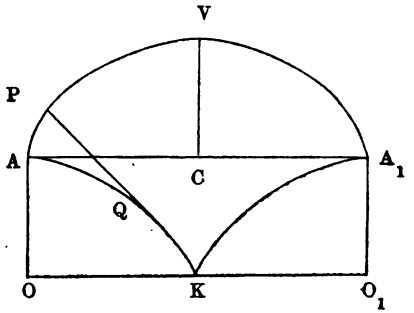
determined, and the whole length $AVA_1 = 4$ diameters of the circle.

Also the length of any part, as $VP =$ twice the corresponding chord VQ.

Further, the tangent at P is parallel to the chord VQ.

218. The cycloid has also this remarkable property, that its *evolute* is another equal cycloid.

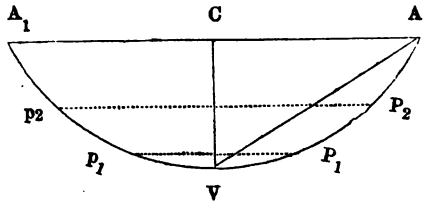
Thus if in fig. $APVA_1$ is a cycloid, its evolute, or the curve off which a string must unwind to trace it, is composed of two semicycloids, each



equal to half the cycloid, but lying as in fig. with the vertices at A, A_1 and their bases in a line OKO_1 parallel to AA_1. The two semicycloids meeting necessarily as at K.

219. This curve possesses the following mechanical properties.

PROPOSITION. *If a body descend by gravity down the arc of an inverted cycloid lying as in fig. the time of descent to the lowest point is the same from whatever point the motion begins.*



Thus, the time down P_1V = the time down P_2V,
= the time down arc AV,
P_1, P_2 being any points in the curve.

This property is easily verified by experiment. If a grooved curve be cut in the form of a cycloid, and several balls be let fall from different points in the groove at the same moment, they will arrive at the lowest point together.

220. Also if a single ball be let fall down the curve, it will when it arrives at the lowest point have acquired a velocity which is sufficient to carry it up a continuation of the curve on the other side to a height equal to that through which it had descended. If, for instance, it began to descend from P_1 it will ascend to p_1, if from P_2 it will ascend to p_2, &c. When it has reached this

highest point on the other side, it will begin to descend again and return to the point from which it started, and so *vibrate* backwards and forwards continually. In practice the ball will not rise quite to the opposite point on the other side, nor will it return to the same point again: friction and the resistance of the air will gradually diminish the arc of vibration, and cause the motion to cease. But if the times of vibration, or the times from the highest point on one side to the highest point on the other, be observed, they will be found *quam proximè* equal to one another. In vacuo and without friction they must be strictly equal.

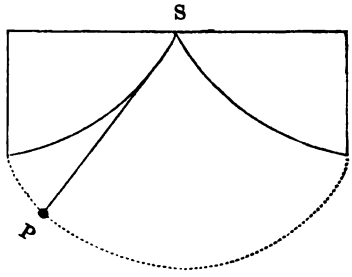
This property gives the cycloid the name of the *Isochronous curve.*

221. PROPOSITION. *If a body descend by gravity from one point to another in the same vertical plane, but not vertically under, it will arrive at the lowest point sooner if it descend in the arc of a cycloid than if it take any other path.*

This remarkable property gives this curve the name of the *Brachystochronous curve.* It is a property of little practical utility, and the aid of the Calculus of Variations is required to establish

it. It is capable however of experimental illustration. If we take the grooved curve mentioned above, and a straight plane from the highest point to the lowest as the chord AV in the figure, it will be found that if two balls are let fall from A at the same moment, and one travels by the curve while the other takes the straight path, the one which moves along the curve will arrive at the lowest point before the other.

222. The property of the cycloid, mentioned in art. 218, enables us to exhibit the isochronous descents by a very simple arrangement. Suspend a heavy ball by a flexible cord between two semicycloidal cheeks, as in fig. and make the length of the cord equal to one of the semicycloidal arcs, then, raise the ball to the highest point by winding the cord on the cheek, it will, when let go, describe by its evolution an equal semicycloid, and having acquired a velocity at the lowest point, it will coil the string on to



the opposite cheek, and so be made to continue in the other half of the same cycloid. When its motion is destroyed in the ascent, it will begin to descend again, and so continue vibrating, and the times of vibration will be the same whether it vibrate in a large or small arc.

223. We have in fact here a cycloidal pendulum, and it is evident from the above, that such a pendulum would always keep the same time, although the arc of vibration may from any cause be increased or diminished. Huygens proposed to take advantage of this property, and to apply such pendulums to clocks or time keepers: but it has been found that the inconveniences resulting from the complicated apparatus necessary to introduce the cycloidal motion are more than equivalent to the advantage of perfect theoretical isochronism. Partially however the cycloidal motion has been of late introduced in some more accurate constructions.

224. PROPOSITION. *The absolute time of descent of a pendulum from any point in the curve of a cycloid, to the lowest point, bears to the time in which a heavy body would fall by gravity through*

a space equal to half the length of the thread, the same ratio as half the circumference of a circle bears to its diameter.

This proposition cannot be established without mathematical formulæ, but the conclusions which follow from it are of considerable importance. The proposition itself may be experimentally verified in the following manner. The ratio of the circumference of a circle to its diameter is expressed approximately by the fraction $\frac{22}{7}$, or by the square root of 10. If this latter approximation be taken, it will follow, that the time of descent in the arc of the cycloid nearly equals the time required for the fall of a heavy body through $\frac{5}{4}$ of the length of the thread. Let then a body be liberated from a point elevated $\frac{1}{4}$ of the length of the thread above the point of suspension at the same moment that the pendulum begins to descend, and the body will meet the pendulum at the lowest point.

Hence we may infer the following

225. PROPOSITION. *The times of vibrations of different pendulums are as the square roots of the length of the strings.*

Thus if we have four pendulums, the lengths of

whose strings are as the numbers 1, 4, 9, 16: the times of vibration are to one another as 1, 2, 3, 4.

The truth of the proposition may be inferred from the preceding; for since, by art. 180, the times of falling through different spaces are as the square roots of the spaces, the times of vibrations of different pendulums will be as the square roots of their lengths. It is also capable of experimental verification.

226. It follows also from art. 224, that the exact time of vibration for a cycloidal pendulum is expressed by the formula

$$t = \pi \sqrt{\frac{l}{g}},$$

where t = the time of vibration;

l = the length of the pendulum;

π = 3.14159.... or the ratio of the circumference of a circle to its diameter;

g = the force of gravity at the place of observation;

= 32.1908 feet in the latitude of London.

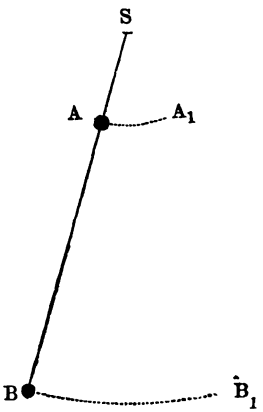
227. It has been remarked, art. 223, that the cycloidal pendulum is inapplicable in practice; but it is evident from the figure and mode of de-

scription of the cycloid, that the curvature of a small portion of the arc near the vertex is very nearly constant, and the same as that of the arc of a circle whose radius is equal to the length of the string. In fact, if a ball is simply suspended to a point by the side of the point to which a cycloidal pendulum is attached, and if the lengths of the strings are the same, the vibrations of the two will be found to correspond as long as the arcs of vibration do not exceed three or four degrees. If the arc of vibration in the case of the one simply suspended, and which necessarily moves in a circle, be increased beyond this limit, there will be an increase in the time of vibration; but within this, the vibration in a small circular arc may be regarded as identical with the vibrations in the arc of a cycloid, and a circular pendulum follows the same laws as a cycloidal. The above conclusions apply therefore to the common pendulum as long as the arc of vibration is small.

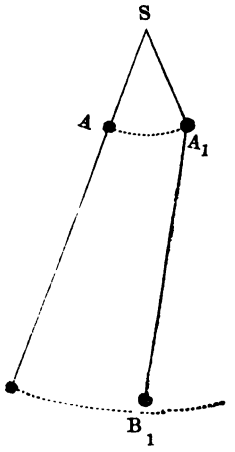
228. In the above statements, the weight as well as the inertia of the pendulum has been referred entirely to one point; we have traced the motion of the ball as the motion of a single point, and have not taken into account the weight

of the string. In practice, the pendulum consists of many parts, and these are not all at the same distance from the point of suspension. This circumstance will have an effect upon the time of vibration, as may be seen in the following experiment.

Suppose two equal balls A and B be hung on the same string SAB attached to a point of suspension S, and let the length SB = 4SA. If this string be drawn out of the vertical, as in fig. the ball A would if free vibrate in the arc AA_1, and the ball B in the arc BB_1; but as the lengths of the strings by which they are suspended are as 4 : 1, the times of the vibrations of the balls if free would be as 2 : 1, by art. 225; hence B could no arrive at B_1 until A had returned to its original position; or, when A is at A_1, B will be at its lowest point; it follows from this, that B will pull back A and retard its motion, while A will tend at first to accelerate B, and the string will



no longer present a continuous straight line, but bent as in fig.



But if instead of a string the balls be fastened on a rigid rod which cannot bend, both must vibrate with a common motion; then the effect will be, that the body nearest the point of suspension will have a tendency to accelerate the vibration, while that farthest off will retard.

This remark may be extended to any number of bodies thus connected. Suppose several thus attached to a rod, as in fig. those nearest the point of suspension S will have an accelerating effect, and will be endeavouring to vibrate in a shorter time, while those more remote will retard; but among them all, there will be found a certain point which will separate those which are moving faster than their natural rate from those which are moving slower; or a point which if suffered to vibrate by itself would perform its vibrations in the same time as the

whole system does. Such a point is of great importance in the motion of pendulums, and is called the *centre of oscillation*, and it may be thus defined.

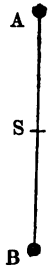
229. DEFINITION. *The centre of oscillation of a pendulous body is that point in which, if the whole mass could be concentrated and suspended by a string without weight, the vibrations of such an imaginary pendulum would be performed in the same time as in the compound one.*

When then we speak of the length of a pendulum, the distance of the centre of oscillation from the point of suspension is always meant, whatever be the form of the body, and the conclusion and formula before deduced apply for this length.

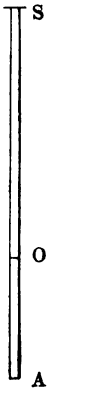
230. The position of the centre of oscillation cannot be determined, even in bodies of a symmetrical form, without the higher mathematics. A few conclusions may be mentioned here, in order to shew that this point is not in the least identical with the centre of gravity. In the case of art. 228, where A and B are two equal balls, and SB = 4 SA, (fig.) the centre of oscillation is at some point O, so that OS = $3\frac{2}{7}$ of SA, and not midway between A and B.

If the point of suspension be between two equal balls, connected by a rigid rod without weight, as in fig. and if $SA = 1$, $SB = 3$, the centre of oscillation is at an imaginary point beyond B, and so that $SO = 5$.

By placing the point of suspension in this case nearer to the centre of gravity, the point O, or the centre of oscillation, is removed to a greater distance, and such a pendulum would consequently vibrate very slowly, or like a very long one. If S were exactly at the centre of gravity, (fig.) the system could not vibrate at all, for by art. 80, it would remain at rest in all positions.



If a simple rod be made to vibrate about one end, its centre of oscillation will be at a distance of $\frac{2}{3}$ of its length; or $SO = \frac{2}{3} SA$. (fig.)



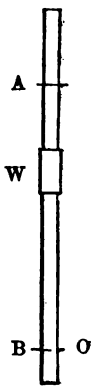
231. If a body be suspended by an axis instead of a simple point, there will be several points at the same distance from this axis, each of which will be a centre of oscillation, and a line drawn at this distance parallel to the axis of

suspension in the plane containing this axis and the centre of gravity will determine the position of all these points, or the position of what may be called the axis of the centres of oscillation.

232. PROPOSITION. *The axis of the centres of oscillation and the axis of suspension are reciprocal or interchangeable.*

This important and remarkable proposition is easily verified by experiment. Invert a pendulum, and suspend it by its centre of oscillation: its former point of suspension becomes its new centre of oscillation: for the time of oscillation of the pendulum will be the same in either position.

233. This property was applied by Captain Kater for the accurate determination of the length of the pendulum which will vibrate seconds in the latitude of London. For this purpose a rod of a certain length is furnished with knife edges or triangular plates of steel at two points, as A and B in fig. B having been roughly ascertained to be very near the centre of oscillation when the pendulum is made to vibrate about A. The knife edges



at one of the points, as, for instance, A, are made to rest by their angles on agate plates, so that the pendulum may oscillate freely. The time of oscillation is observed while the pendulum is in this position, and then it is reversed, so that the knife edges at B rest on the agate plates. It is made to oscillate again, and if the time of oscillation is the same as before, then B is exactly at the centre of oscillation. But if it is not the same, a sliding weight W with which the rod is furnished is moved, until the times become precisely the same, whether the pendulum vibrates about A or about B. When this is the case, the distance AB gives the length of a simple pendulum, which would vibrate in the same time, and from this known length, the length of that which would vibrate seconds can be found by simple calculations from art. 226.

The length of the pendulum which would thus vibrate seconds in the latitude of London is 39.1392 inches, and to this as a standard all our measures of length are referred.

The calculation of the seconds pendulum are made for a given temperature, viz. 62^0, and for a vacuum.

233. Temperature, as we have remarked, art. 31, affects the dimensions of all bodies. The materials of which a pendulum is made expand and contract by increase or decrease of heat. The pendulum therefore is continually increasing or decreasing in length, and hence there arises a change in the *time* of oscillation. Hence in clocks which are required to keep accurate time, it is necessary to introduce some contrivance for *compensating* this variation. There are various kinds of *compensation* pendulums, the general principle of which is to make the pendulum of two parts, one of which shall expand upwards, while the other expands downwards by the addition of heat.

234. The pendulum furnishes not only a convenient measure of time and a standard of length, but observations on its vibrations furnish us with other important verifications and conclusions. For instance:

The pendulum verifies the fact, that all bodies are equally accelerated by the force of gravity. (See art. 188.) This it does, because a pendulum of lead, iron, ivory, wood, &c. will vibrate in the same time if its length be the same: and

gravity is the cause of the vibration after the original impulse has been given.

235. Again, the pendulum furnishes the means of calculating the force of gravity, or the value of g, which we have been hitherto assuming. This value is obtained from the formula of art. 226, viz.

$$t = \pi \sqrt{\frac{l}{g}}$$

from which if l' = the length of the seconds pendulum, and therefore $t = 1$, we have

$$g = \pi^2 l' :$$

now $l' = 39.1392$ inches, and $\pi = 3.14159$, whence $g = 32.19$ feet in the latitude of London.

236. By means of the seconds pendulum, we may by this same formula ascertain, that the value of g is different at different parts of the earth's surface, and thus the distance from the centre of earth is determined, and hence the figure of the earth. And it is found practically, that when the pendulum is carried towards the equator, its time of vibration is longer, while if it is carried towards the pole, the time is diminished. The earth therefore is not spheri-

cal, but of the form of an oblate spheroid, or the form that would be described by the revolution of an ellipse round its shorter axis.

CHAP. VI.

ON CENTRIFUGAL FORCE.

237. We have before observed, art. 167, that in consequence of the great property of inertia, and as an illustration of the first law of motion, a body when compelled to move in a curve has a tendency to fly off from the curve in the direction of a tangent. This tendency is exerted, whether the body move in a curve in consequence of an attractive force, such as gravity exerts when a body in consequence of an oblique impulse describes a parabola, as in Chap. III. or when it is attracted by a centripetal force, as in Chap. IV. or when it is constrained by a groove, or strings, as in Chap. V. In short, in every case when a body moves in a curve, there is a manifestation of this tendency, which is called *Centrifugal Force.*

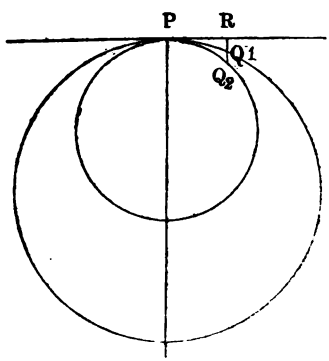
238. The simplest case for consideration is when a body is made to revolve in a circle, by a string attached to a fixed point. In this case the force bears a certain relation to the velocity with which it revolves, and the radius of the circle, as expressed in the following

PROPOSITION. *When a body is compelled to revolve in a circle by any force, the centrifugal*

force is proportional to the square of the velocity, and inversely proportional to the radius of the circle.

It is evident, that the tendency to persevere in a straight line will increase with the velocity, and it is found both by calculation and experiment that it increases as the square of the velocity.

Again, it will be more difficult to turn a moving body into a small circle than a large one, because the deflection from its course is greater in a small than in a large circle. This may be seen by fig. where the deflection Q_2R is greater than Q_1R. Hence *ceteris paribus* the force is greater as the radius of the circle is smaller.



From these two considerations it results that the centrifugal force is measured by the expression

$$\frac{v^2}{r}$$

where v = the velocity of the body,
r = the radius of the circle.

239. In bodies which revolve round a centre, the term *angular velocity* is frequently introduced to express the velocity of rotation independently of the length of the radius.

Angular velocity is measured by *the angle described in a given time*, just as *linear velocity* is measured by the space described in a given time.

Now by Trigonometry,

A circ. arc = angle subtended at centre × rad.

$\therefore \quad v$ = angular velocity × r

$\therefore$ Centrifugal force = (angular velocity)^2 × r.

240. A still more convenient form is obtained by introducing the time of a revolution, or *the periodic time.*

Let t = the time of a revolution or periodic time.

$\pi = 3.14159$, or ratio of circumference of a circle to its diameter.

$\therefore 2\pi r =$ whole circumf. or space described in t.

Hence as generally,

$$s = tv \quad \therefore \ 2\pi r = t \times v$$

$$\therefore \ v = \frac{2\pi r}{t}$$

substituting this in the formula of art. 238, we have,

$$\text{centrifugal force} = 4\pi^2 \frac{r}{t^2}.$$

This formula is convenient, because it is easy to obtain the periodic time by counting the number of revolutions in a given time.

241. The above formulæ do not take into account the mass of the moving body, or, in other words, they apply only when the mass may be taken as unity, or when we are comparing the effects of this centrifugal force on the same body. If different masses are considered, we must multiply by the mass in each case, or take the formulæ

$$\text{centrifugal force} = \text{mass} \times \frac{v^2}{r}$$
$$= \text{mass} \times (\text{ang. vel.})^2 \times r.$$
$$= \text{mass} = 4\pi^2 \frac{r}{t^2}.$$

These formulæ are capable of experimental illustrations by apparatus specially contrived for the purpose by Ferguson. There are also many familiar examples.

242. If the body does not move in a circle, but moves in some other curve by the action of a centripetal force, the tendency must be estimated in the direction opposed to the attractive force.

243. If it is constrained to move upon a given curve, it increases the pressure upon the curve, and the effect of an unit of mass estimated in the

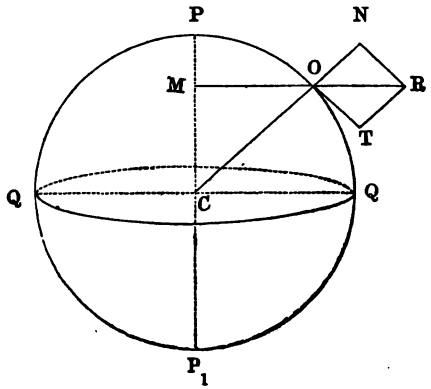
perpendicular to the curve is determined by the formula $\frac{v^2}{\text{rad. of curvature.}}$

When the mass of the moving body is given, this must be multiplied in, as in art. 241.

244. An interesting example of centrifugal force is presented in the diurnal rotation of the earth on its axis. The figure of the earth and the planets, which is that of an oblate spheroid, as mentioned in art. 236, is an effect of centrifugal force exerted before they were in a solid state.

245. In consequence of centrifugal force, the attraction of gravitation is diminished on all bodies on the surface, except at the poles. This effect may be estimated thus:

Suppose PQP_1 (fig.) to represent a section of



the earth, PP_1 the poles, QQ_1 the equator, C the centre, O any place on the surface: and for simplicity suppose the section a circle.

In consequence of the rotation round the axis PCP_1, any point at O has a tendency to fly off in the direction MO produced, for O revolves in a circle parallel to the equator, of which M is the centre and MO the radius. Take any line OR in MO produced to represent this tendency or centrifugal force: this may be resolved into OT, ON, of which ON is employed in diminishing gravity, or the attraction towards the centre. But by sim. tri.

$$\begin{aligned} ON : OR &:: OM &&: OC \\ &:: \sin OCM &&: 1 \\ &:: \text{cos. lat.} &&: 1 \end{aligned}$$

for the angle OCQ = the latitude of the plane $= \lambda$ suppose,

$$\begin{aligned} \therefore ON &= OR \times \cos \lambda \\ &= \text{centrifugal force} \times \cos \lambda \\ &= 4\pi^2 \frac{r}{t^2} \times \cos \lambda \end{aligned}$$

when r = the radius of the circle in which O revolves = MO.

But if R = CO = the radius of the earth,

r or OM = R cos λ.

$\therefore$ The part of the centrifugal force opposed to gravity $= 4\pi^2 \frac{r}{t^2} (\cos \lambda)^2$.

CHAP. VII.

ON THE COLLISION OF BODIES.

246. We have hitherto been examining the phænomena of motion in single bodies, and for the most part as though these bodies were collected in single points; we next proceed to some simple cases of the motions of different bodies acting on each other, and especially those in which one body impinges on another.

247. In all cases of the mutual action of any number of bodies, whether they impinge in consequence of some external force, or whether they approach in consequence of their mutual attraction, there is one general principle of great importance which universally holds, and this is expressed in the following

PROPOSITION. *The centre of gravity of any system of bodies is not affected by any reciprocal or mutual action of the bodies composing the system.*

Hence if this point was at rest before, it will continue at rest although the bodies may approach in any way or impinge. If it was moving before,

it will continue to move in the same way, and it will obey the first law of motion, as though it were a simple moving point. This principle is known by the name of the *conservation of the motion of the centre of gravity:* and we have a grand instance in the system of the universe to which this our planet belongs. The centre of gravity of our solar system (and it must be true also of every other system) is uninfluenced by the mutual attractions and perturbations of the component parts. Observations shew that it is not at rest, it is therefore moving uniformly forward in space unimpeded by the irregularities of the planets; or if, as some suppose, our system is only as it were a satellite to some greater centre, the motion of the whole system round that centre may be determined by the motion of its centre of gravity.

This principle applies also in the same way to the motion of the Earth, accompanied as it is by its Moon, round the Sun. The common centre of gravity of the two moves round the Sun undisturbed by their reciprocal attractions.

248. It is obviously incorrect when we extend this principle to the whole universe to speak of the centre of gravity, for we have restricted the term to the attraction of the Earth: it would be

more correct to call the point the centre of inertia. The centre of inertia by the first law of motion is a centre of parallel forces, (art. 72.) and its position exactly coincides with the position of the centre of gravity.

249. From this principle it follows, that if two or more bodies approach in consequence of their mutual attractions, they would meet in their centre of gravity.

250. When bodies come into collision or impinge on one another, the effects which are produced depend on the form, weight, and nature of the bodies, and especially on the degree of their elasticity. In order to simplify the question, it is usual to consider the bodies at first as entirely without elasticity, and also of such a nature that they shall continue in contact and move on as one body after they have impinged. Although no such bodies exist, yet this is not an unphilosophical way of proceeding, because we can then separately determine the effects which are due to a different condition.

251. We begin also with the simplest case of two such *perfectly inelastic* bodies moving in the same direction with different velocities, so that one overtakes the other.

In this case, when the latter overtakes the first, it will impart to the first some of its own velocity, the velocity of the first will increase and of the other decrease, until having acquired a common velocity they will move on together. By the third law of motion (art. 177.) there can be no loss of momentum, hence the momentum before and after impact must be the same.

Let then m_1, m_2 represent the masses of the bodies

v_1, v_2 represent their respective velocities before impact

x represent their common velocity after.

$\therefore$ the momentum of the two

$$\textit{before} = m_1 v_1 + m_2 v_2$$

$$\textit{after} = (m_1 + m_2)\, x.$$

Equating these we obtain

$$x = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

252. If the bodies are moving in opposite directions, the same law applies; only in this case, the momentum *after* impact is obtained by the difference of the momenta instead of the *sum*, and

$$x = \frac{m_1 v_1 - m_2 v_2}{m + m}$$

This formula assumes, that the momentum of m_1 is greater than of m_2, and consequently the bodies will move together after impact in the direction in which m_1 was moving before; but if the momentum of m_2 is the greater, the value of x will have a negative sign, and they will move in the other direction after impact.

253. If the momenta are equal, $x = 0$, and they remain at rest after impact.

254. If now we suppose the bodies *elastic*, we introduce a force which separates them after collision, and the amount of this force of separation depends on the degree of elasticity. If, as before, we take the case of the one body overtaking the other, this force of elasticity increases in the *overtaking* body the velocity lost, and it may happen that it takes so much from its velocity as to cause it to recede, while it increases in the *overtaken* body the velocity gained.

In order therefore to determine the effect of the collision, we assume that the striking body *loses more*, while the struck body *gains more*, and the additional quantity thus lost and gained depends on the elasticity.

Those bodies are called *perfectly* elastic, where this additional quantity would be equal to the

former: if this quantity is half the former, they are called half-elastic, and so on.

255. There are no known bodies which are perfectly elastic; the degree of elasticity is therefore represented in general by a fraction which is called *the modulus of elasticity*. This may be denoted by the symbol λ, and the calculation of the motion of the bodies is conducted thus, if x represents, as in art. 252, the *common* velocity, if the bodies are inelastic.

Then by impact, if inelastic,

$$\text{the loss of } m_1 = v_1 - x;$$

but by its elasticity it loses also a portion

$$= \lambda (v_1 - x);$$

$$\therefore \text{ whole loss of } m_1 = (v_1 - x) + \lambda (v_1 - x)$$
$$= (1 + \lambda) (v_1 - x);$$

and similarly

$$\text{whole gain of } m_2 = (1 + \lambda) (x - v_2).$$

If the bodies move in opposite directions, or meet, it is only necessary to change the signs, as in art. 252.

256. These formulæ are capable of experimental verification, by suspending balls by equal strings, so that when put into motion they move in arcs of the same circle, and meet *directly*. By such experiments also we determine the value of

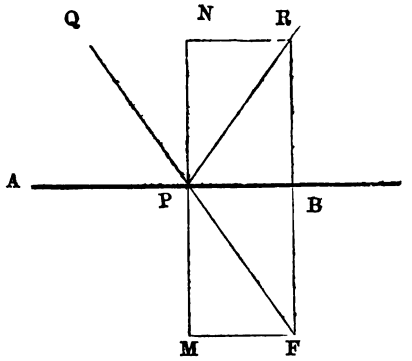
λ in different substances. We can measure the arcs described before and after impact, and thus calculate the velocities before and after. The formulæ above deduced will, if we know the masses, immediately determine the value of λ.

257. The above are all cases of *direct* impact, and it must be assumed that the bodies strike in a direction which passes through the centre of gravity of each. The investigation of the effects of *oblique* impact where the bodies move in different directions, or when the centres of gravity are not in the line of motion, is much more complicated.

258. In the case of the impact of a body on a hard plane, the effect is easily determined. If it impinge in a direction perpendicular to the plane, it will rebound with a velocity depending on the *modulus* or measure of *elasticity*. If it is perfectly elastic, it will rebound with an equal velocity; if half elastic, with half the velocity; and so on.

259. But if it impinge obliquely, it will be *reflected* from the plane in a different direction: and if in fig. APB represent the plane, QP the direction in which the body moves, P being the point of impact, and if PN be drawn perpendi-

cular to the plane, the angle QPN is called the



angle of incidence, then a line drawn on the other side of the perpendicular, as PR, may represent the direction of reflection, and the angle NPR *the angle of reflection*, and this angle will depend on the degree of elasticity, as expressed in the following articles.

260. PROPOSITION. *If a perfectly elastic body impinge on a hard plane, the angles of incidence and reflection are equal, and the velocities before and after impact are equal also.*

Let QP (fig.) represent the direction of motion before impact as before, this motion may be represented by a line PF taken in that direction

produced, and this may be considered as the resultant of two PB, PM, one parallel to the surface, the other perpendicular to it. Now since the ball is perfectly elastic, the component PM will cause a rebound in the direction PN and equal to PM. If then we measure PN = PM, the ball after impact will be affected by the two motions PB and PN, and these two together will cause it to move in the diagonal of the parallelogram NPBR: and it is easy to prove that the direction of PR, the diagonal of this parallelogram, is such, that the

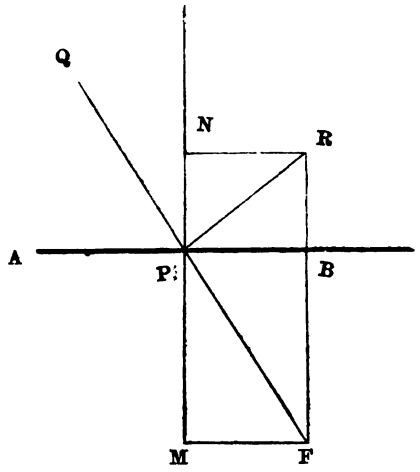
$$\angle NPR = \angle QPN:$$

and also its magnitude is such, that it equals PF, which represented the motion before.

261. PROPOSITION. *If an imperfectly elastic body impinge on a hard plane, the tangent of the angle of incidence : tangent of the angle of reflection :: the modulus of elasticity : 1; and the velocity before : vel. after :: sine of the angle of reflection : sine of the angle of incidence.*

This may be proved by a process very similar to that employed in the last, only as the elasticity is not perfect, the component PM will cause a rebound not equal to it, but so that (fig.) PM : PN :: 1 : λ. By measuring PN according to this

ratio and completing the parallelogram, PR represents the direction and motion after impact, and the proposition is easily proved by Trigonometry.



$$\text{For tan. inc.} = \text{tan. MPF} = \frac{MF}{PM}$$

$$\text{tan. refl.} = \text{tan. RPN} = \frac{NR}{PN} = \frac{MF}{PN}$$

$$\because \text{ by div. } \frac{\text{tan. inc.}}{\text{tan. refl.}} = \frac{PN}{PM} = \lambda.$$

$$\text{also } \frac{\text{vel. before}}{\text{vel. after}} = \frac{PF}{PR} = \frac{\text{sin. PRF}}{\text{sin. PFR}} = \frac{\text{sin. RPN}}{\text{sin. FPM}} = \frac{\text{sin. refl.}}{\text{sin. inc.}}$$

Both these propositions are capable of many experimental illustrations.

CHAP. VIII.

ON THE MOTION OF A SYSTEM.

262. THE determination of the motion of any system of bodies requires the aid of high mathematical calculations, but many of the conclusions are of extreme importance, and capable of experimental verification.

One great principle, the conservation of the motion of the centre of gravity, was enunciated in the last chapter.

263. From this principle it follows, that if an impulse be communicated to a body in a direction not passing through its centre of gravity, there will be a complex effect produced; part of the impulse will produce a *progressive motion* of the centre of gravity, which will obey the first law of motion; the other part will produce a *rotation* of the body about that point.

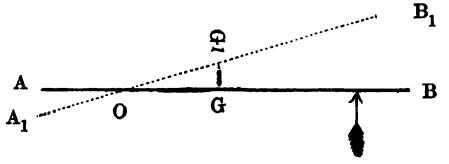
These two motions of *translation* and *rotation* are, when thus produced by one and the same impulse, independent of each other; the motion of translation may be destroyed, while the rotation will continue as before.

Illustrations of this are seen in the motion of a ball thrown from the hand, of a billiard ball, &c.

264. But the grandest illustrations are seen in the motions of the sun and planets. One impulse imparted to all the bodies of our system their motions of rotations round their axes, and their motions in their orbits, and the same impulse which imparted the rotation, would in all cases have communicated a motion of translation, unless a force were applied to neutralize this part of the effect. This consideration gives another reason for assuming that our sun has a motion of translation. The spots on its surface indicate its rotation, the impulse which caused it to rotate would, if not checked, have given also a progression of its centre of gravity.

265. In general, when several forces act on a body, or when the body be not symmetrical, the rotation does not at once settle down into a steady and permanent, or stable, rotation as round a fixed axis, and it is interesting to watch the changes of rotatory motion which a body undergoes. At the very beginning of the motion, while only a small angle has been described by any part of the rotating body, there will be certain points all lying on the same straight line, which

will for a time have no motion, but will remain in their original position, while the centre of gravity about which the rotation really takes place moves a little way forward. This may be seen by the fig. in which AOB represents a



straight line in any body passing through G its centre of gravity. Now if an impulse be given to the body as indicated by the direction of the arrow, the first result will be, that G will move forward, and the line will take a position as indicated by the dotted line A_1OB_1, so that the point O has no motion whatever; now this will also be the case with a number of points all lying in a line through O; and if we suppose the impulse given entirely in the plane of the paper, and AOB to represent a section of a plane at right angles to the plane of the paper, all these points will lie in this plane, and will give what is called an *axis of Spontaneous Rotation.*

This, which we have here shewn in a simple

case only, will be the same in more complicated cases.

A simple verification is obtained both of this and the foregoing principle, by giving a fillip to a pen or light strip of wood as it lies on a table.

266. This axis of spontaneous rotation sometimes coincides with the axis of the centres of oscillation, art. 231, but its position depends on the forces acting.

The position of this axis explains why it is easier to balance a long pole on one end than a short one.

267. We have observed, that the rotation of a body does not always immediately settle down into a permanent condition. In consequence, however, of the opposite tendencies of the centrifugal forces of the different particles, this generally takes place speedily; and it can be proved, that in every system there are *at least* three axes around which, if the rotation is once adjusted, it will remain *permanent*, unless some disturbance interferes. These three axes are at right angles to one another, and pass through the centre of gravity, and from the property enunciated they are called *principal axes*.

268. In every body or system of whatever form, there are *at least* three such axes, but there may be many more.

Thus in a sphere, every diameter is a principal axis.

In every symmetrical body, the axis of symmetry is one principal axis, and any pair of lines at right angles to one another in the plane which passes through the centre of gravity, and at right angles to the axis, are the other two.

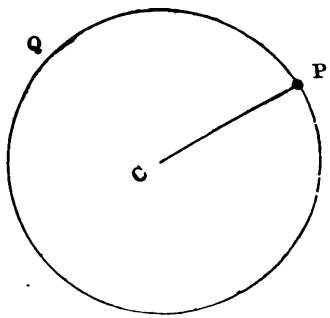
Hence in a spinning top, the axis of symmetry is a principal axis.

In the hoop, one principal axis is an imaginary line through the centre of the circle, and at right angles to the plane.

269. But these principal axes possess another remarkable property, which may be expressed by the introduction of the term *moment of inertia.*

DEFINITION. *If a single particle rotate round a fixed axis by means of an imaginary arm, the product of the mass of the particle, and the square of its distance from the axis of rotation, is called the* MOMENT OF INERTIA OF THE PARTICLE.

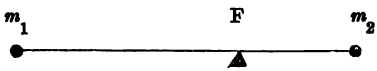
Thus if m = the mass of a particle at P, (fig.) and revolve by means of the arm CP round an axis through C at right angles to the plane of the paper, so that the particle describes the circle PQR, then,



$m \times CP^2$ = the moment of inertia of the particle.

270. DEFINITION. *The moment of inertia of a system is the sum of the moments of all the particles.*

The term moment of inertia is convenient principally to express the results of calculation. We may however conceive the importance of a term to express this product by supposing two particles m_1, m_2 in equilibrium on a lever $m_1 F m_2$, (fig.) the arms of which are r_1 r_2. The statical



moments of m_1 and m_2 are $m_1 r_1$ and $m_2 r_2$ and in equilibrium $m_1 r_1 = m_2 r_2$.

But if now a rotatory motion be communicated, so that the lever revolves in the plane of the paper, the velocities of m_1 and m_2 are respectively proportional (art. 95.) to r_1 and r_2: hence the dynamical moment being the product of the statical moment and the velocity, would be expressed by $m_1 r_1^2$ and $m_2 r_2^2$, or by the product of the masses and the squares of the distances from the axis of rotation.

271. Of the three principal axes of rotation, where there are only three, one is an axis round which the moment of inertia of the system is *the greatest possible;* round another the moment is *the least possible;* and round the third it is of *mean* value: and further, if it rotate round either of the axes where the moment is a *maximum* or *minimum*, the rotation is *stable*, if round the other, it is *unstable*.

272. In bodies where there are several sets of principal axes with one axis common, as in the symmetrical bodies, (art. 269.) this common axis is always one of maximum or minimum moment: and in general where the axis of symmetry is the shortest diameter, the rotation round this is both permanent and stable.

For example, the shortest diameter of an

oblate spheroid, being its shortest principal axis, is that about which, if any rotation be given, the spheroid will tend to rotate, and into a rotation about which it will ultimately settle. We have a striking illustration of this in our system. The planets are all oblate spheroids, and it is about their least diameters that they all rotate.

Hence if any perturbation, whether from a comet or other cause, should interfere with the rotation of any planet, it will nevertheless ultimately return to its present rotation, if the form remain unaltered.

This will also explain the rotation of a common hoop, it rotates round a shortest diameter.

273. If a body or system is made to revolve round a *fixed* axis which is not a principal axis, it undergoes a strain from the unequally balanced centrifugal forces. In order to estimate this effect and the circumstances of the motion, it is usual to determine the position of a point which is called *the centre of Gyration.*

DEFINITION. *The centre of Gyration is that point in a system into which the whole mass must be supposed to be collected, in order that the moment*

of inertia may be the same as it is when the mass is distributed.

The distance of this point from the axis is called the *radius of Gyration.*

274. The determination of this centre facilitates the determination of the position of the *centre of Percussion,* or *that point in a system rotating round a fixed axis to which an impulse should be applied so as to produce no strain on the axis.*

The determination of this point is of importance in the construction of large tilt hammers, &c. Its position cannot be determined theoretically without the higher mathematics, but practically it may be found by suffering the body to vibrate as a pendulum, and observing the time of vibration; for in general it coincides with the centre of oscillation or rather as there may be several centres of oscillation, it is that particular centre of oscillation which is in the line through the centre of gravity and at right angles to the axis of rotation.

In a straight stick rotated round one end, the centre of percussion is at a distance of $\frac{2}{3}$rds the length from that end. (art. 230.)

CHAP. IX.

RESISTING FORCES.

275. We have before (art. 166.) alluded to certain impediments to motion which interfere with the full effects of the forces we have been considering; these impediments are called by the name of *resisting forces*, or simply *resistances*.

These may be included under the definition of force, for they tend to cause a change of motion, but they are *not* forces in the sense in which we have used the term *force* hitherto. They are utterly incapable of imparting motion, or of increasing any existing motion. They can only destroy motion. Sometimes they are regarded as *passive* forces, while forces which impart motion are called *active* forces.

276. Of these *passive or resisting* forces, the most important are *friction*, and the *resistance of fluid media*, such as air or water; and these are continually impeding the motions of bodies at or near the surface of our earth.

277. The effects of friction are a mechanical advantage when merely equilibrium is desired,

but a disadvantage when motion is produced. So that friction in the one case assists, in the other opposes.

For example, in the single moveable pulley, described in art. 134, theory requires for equilibrium a power (P) equal to half the weight (W), and were there no friction, motion would ensue in the direction of P by a slight addition of power, or in the contrary by a small diminution of P.

But if the pulley is subject to friction, as it always is, then the equilibrium will not be destroyed by any small loss of power, or any small increase in the weight; whereas, if it is required to communicate motion, it will be necessary to give a considerable increase to P, in order to overcome the friction.

This may be expressed in all cases in the following manner. Let P represent the power to maintain equilibrium on the machine, and f the addition which must be made to P to enable it to overcome the friction and put the weight in motion, then f will also express the amount by which P must be diminished in order that W may overcome it and descend. Then any power less

[redacted]

283. The laws which
derived exclusively from exp
are two methods by which the
friction are estimated.

In the first method, the body
horizontal table, a cord passe
parallel to the plane, and over
edge of the table; at the end of
is hung, into which weights are
body begins to move. The ra
in the scale to the weight of th
fraction which is called *the coeffi*

284. In the second method, th
upon a plane which is gradually
the body begins to slide or roll,
that the *coefficient of friction* will
tangent of the angle of the plane's inc

285. The following summary
results of the most careful observatio
I. That friction increases,
α. As the pressure or weight is inc
β. As the areas of the surfaces in
are increased.
γ. As the roughness of the surface
creased.
II. That friction may be diminished,

α. By lessening the weight of the body in motion.

β. By reducing the roughness of the surfaces.

γ. By lessening the amount of surfaces in contact.

δ. By converting a sliding into a rolling motion.

ε. By lubricating the surfaces.

286. Another impediment to motion, and one which has a considerable analogy to friction, arises from the rigidity or imperfect flexibility of the ropes or cords employed. This exerts considerable influence, particularly in the working of pulleys. The degree in which this cause affects the motion of machinery had been but partially ascertained, whilst many external circumstances, such as the dampness or dryness of ropes, the quality of their material, the mode in which they are prepared, &c. all which exert great influence, do not admit of being accurately estimated. That which is most important may be thus stated:

The resistance which the stiffness of the rope offers is directly proportional to the weight that acts on it, and also to the thickness of the rope;

and further, it is inversely proportional to the diameter of the wheel or axle about which the rope is coiled; i. e. the greater the weights to be set in motion by means of the machine, and the stronger the ropes, the greater the resistance from the rigidity of the rope, but the greater the wheel, axle, or roller, the less will this impediment be.

287. The other resisting force, viz. that which arises from air or water, is of universal importance. Nearly every motion takes place in one or other of these fluids, and a moving body must thrust before it or move aside out of its path those particles which it meets. From this source there arises a perpetual loss of motive power.

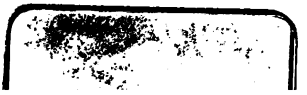
288. The resistance of fluids more properly comes within the province of Hydrostatics, and the subject will be more fully discussed when we treat of fluids in Part II. At present it may suffice to state, that the resistance of fluids depends,

α. On the density of the medium. Thus water is about 800 times denser than air, therefore the resistance of water is 800 times that of air, all other circumstances being the same.

β. On the form and magnitude of the body, more especially on its *frontage*.

γ. On the velocity of the moving body, since the resistance increases as the square of the velocity. This however is only true within certain limits of the velocity. With very high velocities the law is different.

The investigation of these laws will be given when we enter upon the properties and action of fluids.

END OF PART I.

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